Scalar products on semi-vector spaces

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Workshop on QHA 2024

August 8, 2024

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Fulsche and Galke (2023), Werner (1984)

$$\rho(x)\rho(y) = m(x,y)\rho(x+y)$$

Schlather (2024)

$$\exp(i\,[\![x]\!])\exp(i\,[\![y]\!]) = \exp(i\langle x,y\rangle)\exp(i\,[\![x \dot{+} y]\!])$$

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$(G,+)$: locally compact abelian group
$\rho : G \to \mathcal{U}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$ a unitary group of a complex Hilbert space
$m : G \times G \mapsto \{z \in \mathbb{C} : |z| = 1\}$

$(G,\dot{+})$: semigroup
$[\![\cdot]\!] : G \to [0,\infty)$
$\langle\cdot,\cdot\rangle : G \times G \to \mathbb{R}$

Linearity of multiplier/scalar product

Fulsche and Galke (2023), Werner (1984)

$$m(x+y,z)m(x,y) = m(x,y+z)m(y,z)$$

Schlather (2024)

$$\exp(i\langle x \dotplus y, z\rangle)\exp(i\langle x,y\rangle) = \exp(i\langle x, y \dotplus z\rangle)\exp(i\langle y,z\rangle)$$

$(G,+)$: locally compact abelian group
$\rho : G \to \mathcal{U}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$ a unitary group of a complex Hilbert space
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$(G,\dotplus)$: semigroup
$[\![\cdot]\!] : G \to [0,\infty)$
$\langle\cdot,\cdot\rangle : G \times G \to \mathbb{R}$

Key quantities

Schlather (2024)

$$\begin{aligned} \exp(i\,[\![x]\!])\exp(i\,[\![y]\!]) &= \exp(i\langle x,y\rangle)\exp(i\,[\![x \dot{+} y]\!]) \\ [\![x]\!] + [\![y]\!] &= \langle x,y\rangle + [\![x \dot{+} y]\!] \end{aligned}$$

1. Entropy quite similar to a norm
2. Entropies to construct scalar products
3. Examples of entropy: Shannon entropy, $\mathrm{Var}(X)$, $\|x\|^2$, probability measures

Semigroup $(G, \dotplus)$

Entropy

$[\![.]\!] : G \to [0, \infty)$, continuous

G_0: set of deterministic systems, i.e. $[\![\varepsilon]\!] = 0 \quad \varepsilon \in G,\ G_0 \neq \varnothing$

Concatenation $\circ$ (joining as if they were independent)

$[\![\xi \circ \eta]\!] := [\![\xi]\!] + [\![\eta]\!] \qquad \xi, \eta \in G$

Addition $\dotplus$ (joining as they are)

We require: $[\![\xi \dotplus \varepsilon]\!] = [\![\varepsilon \dotplus \xi]\!] = [\![\xi]\!] \qquad \xi \in G, \varepsilon \in G_0$

Semi-metric

Motivating example:

$$\begin{aligned}\mathrm{Var}(X - Y) &= -\mathrm{Var}(X + Y) + 2(\mathrm{Var}X + \mathrm{Var}Y) \\ &= -\mathrm{Var}(X + Y) + 2\mathrm{Var}(X \circ Y)\end{aligned}$$

Require: $\rho(\xi, \varepsilon) = [\![\xi]\!] \qquad \xi \in G, \varepsilon \in G_0$

Semi-metric

$\rho(\xi, \nu) := \rho_a(\xi, \nu) = a\,[\![\xi \dotplus \nu]\!] + (1 - a)\,[\![\xi \circ \nu]\!] \qquad \xi, \nu \in G, a \in \Xi$

$\Xi := \{a \in \mathbb{R} : \rho_a(\xi, \nu) \geqslant 0 \quad \xi, \nu \in G\} \supset [0, 1]$

Comparison to metric

$\rho(\xi, \eta) = \rho(\eta, \xi)$ iff $[\![\xi \dotplus \eta]\!] = [\![\eta \dotplus \xi]\!]$

Semi-scalar product

Geometric motivation: two elements are orthogonal, if the Pythagorean theorem holds, i.e. $[\![\xi \dotplus \eta]\!] = [\![\xi \circ \eta]\!]$

Semi-scalar product: deviation from Pythagorean theorem

$\langle \xi, \eta \rangle := \langle \xi, \eta \rangle_b = b([\![\xi \dotplus \eta]\!] - [\![\xi \circ \eta]\!]) \qquad \xi, \eta \in G$
$\mathrm{b} \in \mathbb{R} \backslash \{0\} \quad \text{s.t.} \quad \langle \xi, \xi \rangle_b \geqslant 0 \qquad \xi \in G$

Comparison to scalar product

For $\xi, \nu, \eta \in G, \varepsilon \in G_0$:

$$\begin{aligned} \langle \xi, \eta \rangle &= \langle \eta, \xi \rangle \text{ iff } [\![\xi \dotplus \eta]\!] = [\![\eta \dotplus \xi]\!] \\ \langle \varepsilon, \varepsilon \rangle &= 0 \\ \langle \xi \dotplus \eta, \nu \rangle + \langle \xi, \eta \rangle &= \langle \xi, \eta \dotplus \nu \rangle + \langle \eta, \nu \rangle \end{aligned}$$

Recall: $m(x+y,z)m(x,y) = m(x,y+z)m(y,z), m : \Xi \times \Xi \mapsto S^1$

Entropy induces scalar product

Construction of Hilbert spaces

$\langle \cdot, \cdot \rangle \rightharpoondown \|x\|^2 := \langle x, x \rangle$

Entropy $\rightharpoondown$ Semi-scalar product

$\langle \xi, \eta \rangle = b([\![\xi \dotplus \eta]\!] - [\![\xi \circ \eta]\!]) \qquad \xi, \eta \in G$

Entropy $\rightharpoondown$ Semi-metric

$\rho(\xi, \nu) = a\,[\![\xi \dotplus \nu]\!] + (1-a)\,[\![\xi \circ \nu]\!] \qquad \xi, \nu \in G, a \in \Xi$
$\Xi = \{a \in \mathbb{R} : \rho_a(\xi, \nu) \geqslant 0 \quad \xi, \nu \in G\}$

real-valued pre-Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$

$[\![x]\!] := \|x\|^2$:

$$\langle x, y\rangle = \frac{\|x+y\|^2 - (\|x\|^2 + \|y\|^2)}{2}$$

i.e. $\langle x, y\rangle = \langle x, y\rangle_H$

$$\rho(x, y) = -\|x+y\|^2 + 2(\|x\|^2 + \|y\|^2)$$

i.e. $\rho(x, y) = \|x-y\|^2$

Square-integrable random variables: Variance

Assume $\mathbb{E}X = 0, X \in H$. $[\![X]\!] = \mathrm{Var}X$.

$$\begin{aligned}\mathrm{Cov}(X, Y) &= \frac{\mathrm{Var}(X+Y) - (\mathrm{Var}X + \mathrm{Var}Y)}{2}\\ \mathrm{Var}(X-Y) &= -\mathrm{Var}(X+Y) + 2(\mathrm{Var}X + \mathrm{Var}Y)\end{aligned}$$

Shannon entropy

α : finite alphabet, transmitted letter $X \sim p_X$ given by p_x, $x \in \alpha$.

$$[\![X]\!] = -\sum_x p_x \log p_x$$

word of length 2: assume independent letters
$\rightharpoondown$ concatenation semigroup $(G, \circ)$
second letter $Y \sim p_Y$

$$\begin{aligned}[\![X \circ Y]\!] &:= -\sum_x \sum_y (p_x p_y) \log(p_x p_y) \\ &= [\![X]\!] + [\![Y]\!]\end{aligned}$$

X, Y dependent

X, Y with discrete, joint probability density p_{XY}
p_X, p_Y : margins of joint probability density
$[\![\cdot]\!]$ = Shannon entropy

$$X \dotplus Y := Z, \qquad Z \sim p_{XY}$$

$$\begin{aligned} \langle X, Y\rangle &= -[\![Z]\!] + [\![X]\!] + [\![Y]\!] \\ \rho(X, Y) &= 2\,[\![Z]\!] - ([\![X]\!] + [\![Y]\!]) \end{aligned}$$

$\langle X, Y\rangle = I(X, Y)$ 'mutual information'
ρ is an ordinary metric: 'variation of information' (Meilă, 2007)

$I(X, Y) \geqslant 0$: simultaneously semi-metric and semi-scalar product

Lebesgue measure

$G := \{\text{compact sets in } \mathbb{R}^d\}$
$\dot{+} := \cup$
$[\![A]\!] := \lambda(A)$

$$\begin{aligned}\langle A, B\rangle &= -[\![A \cup B]\!] + ([\![A]\!] + [\![B]\!]) \\ &= [\![A \cap B]\!]\end{aligned}$$

$$\begin{aligned}\rho(A, B) &= 2\,[\![A \cup B]\!] - ([\![A]\!] + [\![B]\!]) \\ &= [\![A \cup B]\!] - [\![A \cap B]\!] \\ &= [\![A \Delta B]\!]\end{aligned}$$

Outline

Transformation of systems

Now: G set of transformations/ family that is stable under some kind of transformation

Motivating example: Gaussian random variables
Until now: $X + Y$
Now: λX

$$\begin{aligned} G &= \{X \sim \mathcal{N}(0,\sigma^2) : \sigma \in \mathbb{R}_0^+\} \\ &= \{\sigma X_1 : \sigma \in \mathbb{R}_0^+, X_1 \sim \mathcal{N}(0,1)\} \end{aligned}$$

$\Rightarrow$ identify G with $\mathbb{R}_0^+$
Modelling perfectly dependent Gaussian random variables:

$$\begin{aligned} + : (\sigma_1, \sigma_2) &\mapsto \mathrm{Var}(\sigma_1 X_1 + \sigma_2 X_1) = \mathrm{Var}((\sigma_1 + \sigma_2) X_1) \\ &= (\sigma_1 + \sigma_2)^2 \mathrm{Var} X_1 = (\sigma_1 + \sigma_2)^2 = [\![\sigma_1 + \sigma_2]\!] \\ \cdot : (\sigma_1, \sigma_2) &\mapsto \mathrm{Var}(\sigma_2 \sigma_1 X_1) = \sigma_1^2 \sigma_2^2 = [\![\sigma_1]\!] \, [\![\sigma_2]\!] \end{aligned}$$

Semiring $(G, +, \cdot)$

Rescaling: performing 2 trafos in a row does not create additional noise/uncertainty

Rescaling

$\xi \in G$ s.t. $[\![\xi\nu]\!] = [\![\xi]\!]\,[\![\nu]\!] \quad \nu \in G$

Scale-invariance

Every $\xi \in G$ is a rescaling $\rightharpoonup$ G and its entropy are scale-invariant

G scale-invariant

$\langle \xi, \xi \rangle = 0 \Leftrightarrow [\![\xi]\!] = 0$
$\langle \xi\nu, \eta\nu \rangle = [\![\nu]\!] \langle \xi, \eta \rangle$

Semiring & units

Motivation:

Physical units

$3kg, 5m, \ldots$: pairs of scalar & unit connected by 'glueing' together
formally: $\cdot : G \times G_u \to ? \quad (x, u) \mapsto xu$
G : semiring $(G, \dot{+}, \cdot)$
G_u: group $(G_u, \cdot)$

Special case: $G \times G_u \subset G \times G$

- $G_u \subsetneq G$: standard notion of units
- $G_u = G \backslash \{0\}$: G semifield

Halfaxes in $\mathbb{C}$

$G = \mathbb{R}_0^+$
$G_u = \{\exp(k\pi/4) : k = 0, \ldots 3\}$

Units

Semiring $(G, \dot{+}, \cdot)$, group $(G_u, \cdot)$

$$[\![\cdot]\!] : G_u \mapsto \mathbb{C}\backslash\{0\} \text{ s.t. } [\![u_1 u_2]\!] = [\![u_1]\!]\,[\![u_2]\!] \quad u_1, u_2 \in G_u$$

$$[\![xu]\!] := [\![x]\!]\,[\![u]\!] \quad x \in G, u \in G_u$$

Extension of entropy

$\xi u \ddot{+} \eta u := (\xi \dot{+} \eta) u \qquad \xi, \eta \in G, u \in G_u$
If extension to arbitrary $\xi u_1 \ddot{+} \eta u_2$ exists:

$$[\![x_1 u_1 \ddot{+} x_2 u_2]\!] := [\![x_1 u_1]\!] + [\![x_2 u_2]\!] + \langle x_2 u_2, x_1 u_1 \rangle$$

$x_j \in G,\ u_j \in G_u$

Implying:

$\langle x_1 u_1, x_2 u_1 \rangle = [\![u_1]\!] \langle x_1, x_2 \rangle \qquad x_1, x_2 \in G,\ u_1 \in G_u$

$x_j \in G, u_j \in G_u$

Problem: $\langle x_1 u, x_2 u \rangle = [\![u]\!] \langle x_1, x_2 \rangle$, desired: $\langle x_1 u, x_2 u \rangle = \langle x_1, x_2 \rangle$

Transfer semi-scalar product/entropy w.r.t. u_1:

$$\langle x_1 u_1, x_2 u_2 \rangle_{\mathrm{u}_1} := \langle x_1, x_2 u_2 u_1^{-1} \rangle$$

$$\left[\!\!\left[\ddot{\sum_j} x_j u_j \right]\!\!\right]_{\mathrm{u}_1} := \left[\!\!\left[\ddot{\sum_j} x_j u_j u_1^{-1} \right]\!\!\right]$$

Implying:

$$\langle x_1 u_1, x_2 u_2 \rangle_{\mathrm{u}_1} = [\![u_1]\!]^{-1} \langle x_1 u_1, x_2 u_2 \rangle$$

$$\left[\!\!\left[\ddot{\sum_j} x_j u_j \right]\!\!\right]_{\mathrm{u}_1} = [\![u_1]\!]^{-1} \left[\!\!\left[\ddot{\sum_j} x_j u_j \right]\!\!\right]$$

Extension to homogeneity

Until now: pre-Hilbert space H as a semigroup $(H, \dot{+})$

Motivation: introduce scalar multiplication, improve linearity

$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \rightharpoondown \langle \xi \dot{+} \eta, \nu\rangle_a + \langle \xi, \eta\rangle_a = \langle \xi, \eta \dot{+} \nu\rangle_a + \langle \eta, \nu\rangle_a$

$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle \rightharpoondown ?$

Scalar multiplication as (some kind of) rescaling:

pre-Hilbert space H

$\|\lambda x\|^2 = |\lambda|^2\|x\|^2 \qquad x \in H, \lambda \in \mathbb{R}$

$\langle \lambda x, \lambda y\rangle = |\lambda|^2\langle x, y\rangle$

Compare: given a scale-invariant entropy on a semiring $(G, \dot{+}, \cdot)$

$[\![\xi\nu]\!] = [\![\xi]\!]\,[\![\nu]\!]$

$\langle \xi\nu, \eta\nu\rangle = [\![\nu]\!]\,\langle \xi, \eta\rangle \qquad \xi, \nu, \eta \in G$

$[\![x]\!] = \|x\|^2, x \in H$ but $[\![\lambda]\!], \lambda \notin H$ is undefined

Solution: Semi-vector spaces

Semifield $(K, \dot{+}, \cdot)$

1. $(K, \dot{+})$ abelian semigroup with 0
2. $(K \backslash \{0\}, \cdot)$ group with 1
3. left, right distributive law

Semi-vector space

semifield K, abelian monoid $(V, \dot{+})$, scalar multiplication $\cdot : K \times V \to V$

1. left, right distributive law
2. associative law: $(ab)v = a(bv)$
3. $1v = v$, $1 \in K$ $\qquad a, b \in K, v, w \in V$

Semi-inner product on semi-vector space V, $K \subset \mathbb{R}$

1. $[\![\cdot]\!] : V \mapsto \mathbb{R}_0^+$ on $(V, \dot{+})$
2. $\cdot : K \times V \mapsto V$ is scale invariant, i.e. $[\![kv]\!] = c_k\, [\![v]\!]$, $c_k \in \mathbb{R}_0^+, k \in K, v \in V$

Then:

$$\langle v, w\rangle_b = b([\![v \dot{+} w]\!] - [\![v \circ w]\!]) \qquad v, w \in V$$

$b \in \mathbb{R}\backslash\{0\}$ s.t. $\langle v, v\rangle_b \geqslant 0 \qquad v \in V$

Properties

For $\xi, \nu, \eta \in G, \varepsilon \in G_0$:

$$\begin{aligned} \langle \xi, \eta\rangle &= \langle \eta, \xi\rangle \text{ iff } [\![\xi \dot{+} \eta]\!] = [\![\eta \dot{+} \xi]\!] \\ \langle \varepsilon, \varepsilon\rangle &= 0 \\ \langle \xi \dot{+} \eta, \nu\rangle + \langle \xi, \eta\rangle &= \langle \xi, \eta \dot{+} \nu\rangle + \langle \eta, \nu\rangle \\ \langle \lambda v, \lambda w\rangle &= c_\lambda \langle v, w\rangle \end{aligned}$$

G^n: canonical semi-vector space over G

$(G, \dot{+}, \cdot)$ semifield: $\dot{+} : G^n \times G^n \to G^n$, $\cdot : G \times G^n \to G^n$ componentwise

a) Scale-invariant semiring $(G, \dot{+}, \cdot)$ & $(G, \cdot)$ is a group
b) Semiring $(G, \dot{+}, \cdot)$ with entropy & units $G_u = G$ with entropy

$[\![\cdot]\!]_G : G \to \mathbb{R}_0^+$:

$$
\begin{aligned}
[\![\xi]\!] &= \sum_i^n [\![\xi_i]\!]_G \\
\langle \xi, \eta \rangle &= b_G([\![\xi \dot{+} \eta]\!] - [\![\xi \circ \eta]\!]) = \sum_i^n \langle \xi_i, \eta_i \rangle_G
\end{aligned}
$$

with $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, .., \eta_n) \in G^n$, defines entropy and semi-inner product on G^n, respectively.

Possible Applications

Physical units

G_u partially ordered group, $u \leqslant v$ iff $u^{-1}v \in \mathbb{N}$

Extension to complex-valued scalar products

$\Re(\langle x, y \rangle) = (\|x+y\|^2 - \|x\|^2 - \|y\|^2)/2$

Positive definiteness

One-sided deviation from Pythagorean theorem

Open questions

Entropic uncertainty relations

Our theory may connect entropies and scalar products.

Extension of entropy to matrices

Recall:

$$\begin{aligned} \rho(x)\rho(y) &= m(x,y)\rho(x+y) \\ \exp(i\,[\![x]\!])\exp(i\,[\![y]\!]) &= \exp(i\langle x,y\rangle)\exp(i\,[\![x \dot{+} y]\!]) \end{aligned}$$

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Application to Extreme Value Statistics

Univariate maximum

$G := [0, \infty)$, $\dot{+} := \vee$, $[\![\xi]\!] := |\xi|^\alpha, \alpha > 0$.

$$\begin{aligned} \langle \xi, \eta \rangle &= \xi^\alpha + \eta^\alpha - (\xi \vee \eta)^\alpha = (\xi \wedge \eta)^\alpha \\ \rho(\xi, \eta) &= (\xi \vee \eta)^\alpha - (\xi \wedge \eta)^\alpha \end{aligned}$$

ρ is an ordinary metric.

Multivariate maximum

G^n:

$$\langle \xi, \nu \rangle = \|\xi\|_\alpha^\alpha + \|\nu\|_\alpha^\alpha - \|\xi \vee \nu\|_\alpha^\alpha = \sum_i^n (\xi_i \wedge \eta_i)^\alpha$$

for $\xi = (\xi_i, ..., \xi_n), \eta = (\eta_1, ..., \eta_n) \in G^n$.