QHA and limit operators: a good combination

Robert Fulsche
Leibniz Universität Hannover, Germany

Workshop on QHA 2024

Based on joint projects with Raffael Hagger (CAU Kiel), Franz Luef (NTNU) and Reinhard Werner (also LUH)

Contents

- What are limit operators?
- Applications:
  1. The algebra problem of correspondence theory.
  2. The structure of the quotient $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H})$
  3. Closely related to the previous point: Fredholm theory in $\mathcal{C}(\mathcal{H})$
  4. Wiener's Tauberian theory for operators.

Limit functions

Let G be a locally compact abelian group. For simplicity, we assume that G is second countable. We will usually write the group operation additively. Spaces such as $L^p(G)$ are always considered with respect to some fixed Haar measure.
For an (equivalence class of a Borel measurable) fuction $f : G \to \mathbb{C}$ and $x \in G$ we write $\alpha_x(f)(t) = f(t - x)$. The importance of the following well-known and easily proven fact cannot be overestimated:

Lemma 1

Let $f \in L^1(G)$. Then, $G \ni x \mapsto \alpha_x(f) \in L^1(G)$ is bounded and uniformly continuous.

Limit functions

Lemma 1

Let $f \in L^1(G)$. Then, $G \ni x \mapsto \alpha_g(f) \in L^1(G)$ is bounded and uniformly continuous.

Recall that $L^1(G)' \cong L^\infty(G)$ via the dual pairing

$$L^1(G) \times L^\infty(G) \ni (f, g) \mapsto \langle f, g \rangle := \int_G f(t)g(t)\, dt.$$

Lemma 2

Let $g \in L^\infty(G)$. Then, for every $f \in L^1(G)$, the map

$$G \ni x \mapsto \langle f, \alpha_x(g) \rangle = \langle \alpha_{-x}(f), g \rangle$$

is bounded and uniformly continuous.

Limit functions

Hence, we need to talk about bounded and uniformly continuous functions on G, denoted $\mathrm{BUC}(G)$.

$\mathrm{BUC}(G)$ forms a unital C*-subalgebra of $C_b(G)$. Hence, the maximal ideal space of $\mathrm{BUC}(G)$ (the space of multiplicative linear functionals on $\mathrm{BUC}(G)$, endowed with the weak* topology), denoted $\mathcal{M}(\mathrm{BUC}(G))$, is a compact Hausdorff space. By identifying points in $x \in G$ with functionals of point evaluation δ_x, $\delta_x(f) = f(x)$, $\mathcal{M}(\mathrm{BUC}(G))$ turns into a compactification of G. We will usually write $\mathcal{M} = \mathcal{M}(\mathrm{BUC}(G))$. Through the Gelfand transform, $\mathrm{BUC}(G) \cong C(\mathcal{M})$.

Take away message, if you don't know Gelfand theory:

- There exists a certain compactification $\mathcal{M}$ of G to which every function $f \in \mathrm{BUC}(G)$ can be continuously extended.
- $\mathcal{M}$ is universal with respect to this property.
- $\mathcal{M}$ is *almost* as bad as the Stone-Čech compactification.

We will usually also write $\partial G = \mathcal{M} \setminus G$ for the boundary of the compactification, which is a closed subset of $\mathcal{M}$.

Limit functions

Let now again be $f \in L^1(G)$ and $g \in L^\infty(G)$. As mentioned before, the function $\Phi_{f,g}(x) = \langle f, \alpha_x(g) \rangle$ is bounded and uniformly continuous. Hence, we can extend it to a continuous function on $\mathcal{M}$.

Lemma 3

Let $g \in L^\infty(G)$ and $x \in \partial G$. Then,

$$L^1(G) \ni f \mapsto \Phi_{f,g}(x)$$

is a bounded linear functional on $L^1(G)$.

Since $L^\infty(G) \cong L^1(G)$, there exists a unique $h_x \in L^\infty(G)$ such that $\Phi_{f,g}(x) = \langle f, h_x \rangle$ for all $f \in L^1(G)$. We define the limit function of g at $x \in \partial G$ by:

$$\alpha_x(g) := h_x.$$

Limit functions

Lemma 4

Let $g \in L^\infty(G)$*. Then,* $\mathcal{M} \ni x \mapsto \alpha_x(g)$ *is continuous in* w^* *topology.*

We can achieve better results than this. Using the Arzelà-Ascoli theorem, one can show:

Lemma 5

Let $g \in \mathrm{BUC}(G)$*. Then:*

- $\mathcal{M} \ni x \mapsto \alpha_x(g)$ *is continuous with respect to uniform convergence on compact subsets of* G*.*
- $\alpha_x(g) \in \mathrm{BUC}(G)$ *for every* $x \in \partial G$*.*

Limit functions

In suitable situations, the compactification $\mathcal{M}$ can be replaced by a "nicer" compactification. More precisely: Let $\mathcal{A} \subset \mathrm{BUC}(G)$ be a unital C*-subalgebra which is also α-invariant. Denote by $\mathcal{M}_A$ the maximal ideal space of $\mathcal{A}$. By mapping $z \mapsto \delta_z$, we can still map G into $\mathcal{M}_A$. In general, $G \subset \mathcal{M}_A$ may not be open and the "embedding" may not be injective. We write $\beta_-(g)(x) = g(-x)$ for $x \in G$. β_- leaves $\mathrm{BUC}(G)$ invariant.

Theorem 1

Let $\mathcal{A} \subset \mathrm{BUC}(G)$ as above and $g \in \mathrm{BUC}(G)$. Then, $g \in \mathcal{A}$ if and only if $G \ni x \mapsto \alpha_x(g)$ extends to a continuous map

$$\mathcal{M}_{\beta_-(A)} \ni x \mapsto \alpha_x(g)$$

with respect to uniform convergence on compact subsets.

Limit operators

For the talk, we will restrict ourselves to operators on Hilbert spaces. The methods discussed here extend to a larger class of Banach spaces (a certain class of coorbit spaces).
Let again G be an lca group, still assumed to be second countable for convenience. Let $\rho : G \to \mathcal{U}(\mathcal{H})$ a strongly continuous projective unitary representation of G.
We proceed in complete analogy to the case of functions, that we just discussed. For any (say, bounded) linear operator on $\mathcal{H}$, we denote $\alpha_x(A) = \rho(x)A\rho(x)^*$.
Let $\mathcal{T}^1(\mathcal{H})$ denote the trace class on $\mathcal{H}$. This will play a role analogous to $L^1(G)$.

Limit operators

Lemma 6

Let $A \in \mathcal{T}^1(\mathcal{H})$. Then, $G \ni x \mapsto \alpha_x(A) \in \mathcal{T}^1(\mathcal{H})$ is bounded and uniformly continuous.

Recall that $\mathcal{T}^1(\mathcal{H})' \cong \mathcal{L}(\mathcal{H})$ via trace duality:

$$\mathcal{T}^1(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) \ni (A, B) \mapsto \langle A, B \rangle := tr(AB).$$

Lemma 7

Let $A \in \mathcal{T}^1(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$. Then, the map

$$G \ni x \mapsto \langle A, \alpha_x(B) \rangle$$

is bounded and uniformly continuous.

Limit operators

Having the previous result at hand, we can now play the same game as for functions: Given $A \in \mathcal{T}^1(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{H})$, the function $\Phi_{A,B}(x) := \langle A, \alpha_x(B) \rangle$ can be evaluated at $x \in \partial G$. Since $A \mapsto \Phi_{A,B}(x)$ is a bounded linear functional on $\mathcal{T}^1(\mathcal{H})$, there exists a unique $\alpha_x(B) \in \mathcal{L}(\mathcal{H})$ such that $\Phi_{A,B}(x) = \langle A, \alpha_x(B) \rangle$. Further,

Lemma 8

With the above conventions, the map

$$\mathcal{M} \ni x \mapsto \alpha_x(B) \in \mathcal{L}(\mathcal{H})$$

is continuous in w^* topology (which implies continuity in WOT).

A brief recap of some facts from QHA

In the following, we let G again be a (second-countable) lca group, $\widehat{G}$ the Pontryagin dual and $\Xi = G \times \widehat{G}$ the phase space. On $\mathcal{H} = L^2(G)$ we consider the projective unitary representation of Ξ given by

$$U_z f(t) = \xi(t) f(t-x), \quad z = (x, \xi) \in \Xi,\ f \in L^2(G).$$

With this representation, we make the same constructions as before: $\alpha_z(A) = U_z A U_z^*$ for $A \in \mathcal{L}(\mathcal{H})$. Then, $z \mapsto \alpha_z(A)$ acts continuous on $\mathcal{T}^1(\mathcal{H})$ and w* continuous on $\mathcal{L}(\mathcal{H})$. We consider the following class of operators, which plays the role that is, in some sense, analogous to BUC(Ξ):

$$\mathcal{C}(\mathcal{H}) := \{A \in \mathcal{L}(\mathcal{H}) :\ \|\alpha_z(A) - A\|_{op} \to 0 \text{ as } z \to 0\}.$$

A brief recap of some facts from QHA

For a function $f : \Xi \to \mathbb{C}$ and an operator A on $\mathcal{H}$ we formally set:

$$f * A := \int_\Xi f(z)\alpha_z(A)\, dz.$$

Further, we formally set for two operators A, B on $\mathcal{H}$:

$$A * B(z) := tr(A\alpha_z(PBP)).$$

Here, P is the parity operator $P\varphi(z) = \varphi(-z)$ on $L^2(G)$.
Of course, these expressions are in general not well-defined.

A brief recap of some facts from QHA

With these notions of convolutions, one has the following well-known fact:

Theorem 2

The convolutions map boundedly:

$$\begin{aligned}
* :& L^1(\Xi) \times L^\infty(\Xi) \to \mathrm{BUC}(\Xi) \\
* :& L^1(\Xi) \times \mathcal{L}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) \\
* :& L^\infty(\Xi) \times \mathcal{T}^1(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) \\
* :& \mathcal{T}^1(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) \to \mathrm{BUC}(\Xi)
\end{aligned}$$

In an appropriate sense, the convolutions are commutative and associative.

QHA and limit operators

A key result in QHA is the following theorem:

Theorem 3

$\mathcal{C}(\mathcal{H}) = \overline{\mathcal{T}^1(\mathcal{H}) * \mathrm{BUC}(\Xi)}$

Having this result at hand, one can prove the following fact for limit operators:

Theorem 4

Let $A \in \mathcal{C}(\mathcal{H})$. Then, the map $\Xi \ni z \mapsto \alpha_z(A)$ extends to a map

$$\mathcal{M} \ni z \mapsto \alpha_z(A) \in \mathcal{C}_1(\mathcal{H})$$

which is continuous in SOT.

QHA and limit operators

Theorem 4

Let $A \in \mathcal{C}(\mathcal{H})$. Then, the map $\Xi \ni z \mapsto \alpha_z(A)$ extends to a map

$$\mathcal{M} \ni z \mapsto \alpha_z(A) \in \mathcal{C}(\mathcal{H})$$

which is continuous in SOT.

Idea of the proof: By the previous theorem, it suffices to prove the statement for $A = B * f$ with $B \in \mathcal{T}^1(\mathcal{H})$ and $f \in \mathrm{BUC}(\Xi)$. Further, it suffices to prove the statement for B being of rank one: $B = \varphi \otimes \psi$. A further simplification reduces the statement to the case where $\varphi, \psi \in \mathcal{S}(G)$.

Note that for $z \in \Xi$: $\alpha_z(A) = B * \alpha_z(f)$. Now, for $h \in \mathcal{S}(G)$:

$$B * \alpha_z(f)(g) = \int_\Xi \alpha_z(f)(w) \langle g, U_w \psi \rangle_{L^2} U_w \varphi \, dw$$

QHA and limit operators

$$B * \alpha_z(f)(g) = \int_\Xi \alpha_z(f)(w)\langle g, U_w\psi\rangle_{L^2} U_w\varphi \, dw$$

Note that $w \mapsto \langle g, U_w\psi\rangle$ is rapidly decreasing and $U_w\varphi$ is uniformly bounded, such that $w \mapsto \|\langle g, U_w\psi\rangle U_w\varphi\|_{L^2}$ is rapidly decreasing. When $(z_\gamma) \subset \Xi$ is a net converging to $x \in \partial\Xi$, then $\alpha_{z_\gamma}(f) \to \alpha_x(f)$ uniformly on compact subsets, as we have established earlier. Combining these two things yields:

$$B * \alpha_{z_\gamma}(f)(g) \to B * \alpha_x(f)(g).$$

By standard density arguments, this is enough to prove the theorem.

Applications of limit operator theory in QHA

We want to discuss a few applications that limit operator theory has in the domain of QHA:

1. The algebra problem of correspondence theory.
2. The structure of the quotient $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H})$
3. Closely related to the previous point: Fredholm theory in $\mathcal{C}(\mathcal{H})$
4. Wiener's Tauberian theory for operators.

The algebra problem of correspondence theory

Two closed, α-invariant subspaces $\mathcal{D}_0 \subset \mathrm{BUC}(\Xi)$ and $\mathcal{D}_1 \subset \mathcal{C}(\mathcal{H})$ are said to be *corresponding spaces* if $\mathcal{T}^1(\mathcal{H}) * \mathcal{D}_0 \subset \mathcal{D}_1$ and $\mathcal{T}^1(\mathcal{H}) * \mathcal{D}_1 \subset \mathcal{D}_0$.
Fact: For each closed, α-invariant subspace $\mathcal{D}_0$ there exists exactly one closed, α-invariant subspace $\mathcal{D}_1 \subset \mathcal{C}(\mathcal{H})$ and vice verca (this is Werner's correspondence theorem). This fact, and variations of it, have found several interesting applications in operator theory.
Fact: If $\mathcal{D}_0$ and $\mathcal{D}_1$ are corresponding spaces and $A \in \mathcal{C}(\mathcal{H})$, then $A \in \mathcal{D}_1$ if and only if $\mathcal{T}^1(\mathcal{H}) * A \subset \mathcal{D}_0$.
In his '84 paper, Werner studied several properties that are "preserved" under correspondence. The following question he could not answer:

Question 1 (R. Werner, '84)

Is $\mathcal{D}_0$ a C^*-algebra if and only if $\mathcal{D}_1$ is a C^*-algebra?

To this day, this question is not entirely settled.

The algebra problem of correspondence theory

Here is a partial answer:

Theorem 5 (RF, 2020)

If $\mathcal{D}_0$ is a C^*-algebra, then $\mathcal{D}_1$ is a C^*-algebra.

Before discussing the proof, we make some comments:

- In the same direction of the correspondence, ideals in C*-algebras are also mapped to ideals.
- A partial converse can be obtained for $\Xi = \mathbb{R}^{2d}$ by making use of a suitable semiclassical limit.
- Wu and Zhao showed in 2021 for $\Xi = \mathbb{R}^{2d}$ that $\mathcal{D}_1$ is a Banach algebra whenever $\mathcal{D}_0$ is a Banach algebra.

The algebra problem of correspondence theory

Theorem 5 (RF, 2020)

If $\mathcal{D}_0$ is a C^*-algebra, then $\mathcal{D}_1$ is a C^*-algebra.

Idea of the proof: Let $A, B \in \mathcal{D}_1$. Then, an application of the correspondence theorem shows that $z \mapsto \alpha_z(A)$ extends to a continuous function (with respect to SOT)

$$\mathcal{M}_{\beta_-(\mathcal{D}_0)} \ni x \mapsto \alpha_x(A).$$

The same holds true for $z \mapsto \alpha_z(B)$. Since the shifts of operators are isometric, one can deduce that also $z \mapsto \alpha_z(AB) = \alpha_z(A)\alpha_z(B)$ extends continuously to $\mathcal{M}_{\beta_-(\mathcal{D}_0)}$. From here, it is not hard to deduce that for every $D \in \mathcal{T}^1(\mathcal{H})$:

$$D * (AB) \in C(\mathcal{M}_{\mathcal{D}_0}) \cong \mathcal{D}_0.$$

Therefore, another application of the correspondence theorem yields $AB \in \mathcal{D}_1$, finishing the proof.

The structure of the quotient $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H})$

An application of correspondence theory yields:

Theorem 6

Let $A \in \mathcal{L}(\mathcal{H})$. Then, $A \in \mathcal{K}(\mathcal{H})$ if and only if $A \in \mathcal{C}(\mathcal{H})$ and $\alpha_x(A) = 0$ for every $x \in \partial\Xi$.

This observation is a key fact, allowing for a precise description of the quotient $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Definition 1

A compatible family of limit operators (cflo) is a map $\omega : \partial\Xi \to \mathcal{C}(\mathcal{H})$ satisfying:

1. *ω is continuous in w^* topology (on the operator side);*
2. *For every $x \in \partial\Xi$ and $z \in \Xi$: $\alpha_z(\omega(x)) = \omega(x - z)$;*
3. $\sup_{x\in\partial\Xi} \|\omega(x)\| < \infty$;
4. $\{\omega(x) : \; x \in \partial\Xi\}$ *is uniformly equicontinuous.*

The structure of the quotient $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H})$

We denote the space of all compatible families of limit operators by $\mathfrak{lim}\ \mathcal{C}(\mathcal{H})$. Here is the main theorem on these families:

Theorem 7 (RF '24, RF, F. Luef, R. Werner '24)

1. *Let $A \in \mathcal{C}(\mathcal{H})$. Then, $\omega(x) = \alpha_x(A)$, $x \in \partial\Xi$, is a cflo.*
2. *If ω is a cflo, then there exists some $A \in \mathcal{C}(\mathcal{H})$ such that $\omega(x) = \alpha_x(A)$.*
3. $\mathfrak{lim}\mathcal{C}(\mathcal{H})$ *is a C^*-algebra.*
4. $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \cong \mathfrak{lim}\ \mathcal{C}(\mathcal{H})$.

The structure of the quotient $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H})$

The most important part of the theorem is (2): Given a cflo, we can find an operator in $\mathcal{C}(\mathcal{H})$ admitting this cflo as its limit operators!
Idea of the proof of (2): A cornerstone of the proof is

$$\|B + \mathcal{K}(\mathcal{H})\| \cong \sup_{x \in \partial \Xi} \|\alpha_x(B)\|, \quad B \in \mathcal{C}(\mathcal{H}),$$

which is readily established by some standard C*-algebraic argument.
Let $A \in \mathcal{T}^1(\mathcal{H})$ be a regular operator (i.e., $\{\alpha_z(A) : \ z \in \Xi\}$ spans a dense subspace of $\mathcal{T}^1(\mathcal{H})$). By some standard approximate identity arguments, there exists constants $c_k^\gamma \in \mathbb{C}$ and $z_k^\gamma \in \Xi$ (where $\gamma \in \Gamma$ for some directed set Γ) such that

$$\|\gamma(x) - \sum_k c_k^\gamma \alpha_{z_k^\gamma}(A * A * \gamma(x))\|_{op} \xrightarrow{\gamma \in \Gamma} 0$$

Since the family $\{\gamma(x) : \ x \in \partial \Xi\}$ is assumed equicontinuous, this convergence is uniform in x!

The structure of the quotient $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H})$

$$\sup_{x\in\partial\Xi} \|\gamma(x) - \sum_k c_k^\gamma \alpha_{z_k^\gamma}(A * A * \gamma(x))\|_{op} \xrightarrow{\gamma\in\Gamma} 0$$

Note that $[A * \gamma(x)](0)$ is a continuous function on $\partial\Xi$. By Tietze's extension theorem, this can be extended to a continuous function on Ξ, denoted by f_A. Then, $A * A * \gamma(x)$ is (up to some β_-) the limit operator of $A * f$.

Therefore, by $\|B + \mathcal{K}(\mathcal{H})\| \cong \sup_x \|\alpha_x(B)\|$, the net

$$(\sum_k c_k^\gamma \alpha_{z_k^\gamma}(A * f))_{\gamma\in\Gamma}$$

is a Cauchy net in $\mathcal{C}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, hence converges to some $B + \mathcal{K}(\mathcal{H})$. Now, B is (again, up to some β_-) an operator having $\omega(x)$ as limit operators.

Theorem 8 (RF, R. Hagger '19, RF '24, RF, R. Hagger '24?)

Let $A \in \mathcal{C}(\mathcal{H})$. Then, A is Fredholm if and only if $\alpha_x(A)$ is invertible for every $x \in \partial\Xi$ and $\sup_{x\in\partial\Xi} \|\alpha_x(A)^{-1}\| \leq c < \infty$.

The implication "A Fredholm $\Rightarrow \alpha_x(A)$ invertible with $\|\alpha_x(A)^{-1}\| \leq c < \infty$" is rather straightforward, so the other implication is the important one.

Having the theorem about cflo's at hand, this can be done rather easily: For every $x \in \partial\Xi$ set $\omega(x) = \alpha_x(A)^{-1}$. Since $\alpha_z(\alpha_x(A)) = \alpha_x(\alpha_z(A))$, we obtain that $\alpha_z(\omega(x)) = \omega(x - z)$. By the second resolvent identity,

$$\alpha_x(A)^{-1} - \alpha_y(A)^{-1} = \alpha_x(A)^{-1}(\alpha_x(A) - \alpha_y(A))\alpha_y(A)^{-1},$$

$x \mapsto \alpha_x(A)^{-1}$ is continuous in SOT.

Fredholm theory in $\mathcal{C}(\mathcal{H})$

Finally, by some standard Neumann series argument,

$$\|\alpha_z(\alpha_x(A)^{-1}) - \alpha_x(A)^{-1}\| \leq \frac{\|\alpha_x(A)^{-1}\|\|\alpha_z(\alpha_x(A)) - \alpha_x(A)\|}{1 - \|\alpha_z(\alpha_x(A)) - \alpha_x(A)\|\|\alpha_x(A)^{-1}\|}$$

for all $x \in \partial\Xi$ and z sufficiently small. Hence, uniform equicontinuity follows. This shows that $x \mapsto \omega(x) = \alpha_x(A)^{-1}$ is a cflo. Therefore, there exists some $B \in \mathcal{C}(\mathcal{H})$ such that $\alpha_x(B) = \alpha_x(A)^{-1}$ for every $x \in \partial\Xi$. Now,

$$\alpha_x(AB - I) = \alpha_x(A)\alpha_x(A)^{-1} - I = 0, \quad x \in \partial\Xi$$

such that $AB - I \in \mathcal{K}(\mathcal{H})$.

Fredholm theory in $\mathcal{C}(\mathcal{H})$

The “full” theorem goes as follows:

Theorem 9 (RF, R. Hagger ’19, ’24?)

Let $A \in \mathcal{C}(\mathcal{H})$. Then, A is Fredholm if and only if $\alpha_x(A)$ is invertible for every $x \in \partial\Xi$.

The difference: The condition “$\sup_{x\in\partial\Xi} \|\alpha_x(A)^{-1}\| < \infty$” is redundant. So far, we have not found a way of proving this completely within the toolbox of QHA. Localization techniques from the setting of band-dominated operators are used. For details, see Raffael’s talk.

Wiener's Tauberian theory for operators

The following result was presented on the online workshop on QHA in 2021 (strictly speaking, for $\Xi = \mathbb{R}^{2d}$):

Theorem 10 (F. Luef, E. Skrettingland '21)

Let $B \in \mathcal{L}(\mathcal{H})$ and $c \in \mathbb{C}$. Then, the following two statements are equivalent:

1. *For some regular $A \in \mathcal{T}^1(\mathcal{H})$: $A * B - c \cdot \mathrm{tr}(A) \in C_0(\Xi)$;*
2. *For some regular $f \in L^1(\Xi)$: $f * B - c \cdot \mathrm{tr}(f) \in \mathcal{K}(\mathcal{H})$.*

If those equivalent statements hold true, then they also hold true for every $A \in \mathcal{T}^1(\mathcal{H})$ and every $f \in L^1(\Xi)$.

This result motivated two questions:

- What's the proper operator analogue of Pitt's refinement of Wiener's Tauberian theorem?
- Can the space of all operators $B \in \mathcal{L}(\mathcal{H})$ such that $A * B \in C_0(\Xi)$ for every $A \in \mathcal{T}^1(\mathcal{H})$ be classified?

Wiener's Tauberian theory for operators

In a joint project with F. Luef and R. Werner, we discussed these questions. Much to our surprise, both questions can be answered by the same underlying theory.
We will not present the full details of the results here, only the most important cases. Before doing so, we will first talk a little about the classical form of Wiener's Tauberian theorem.

Theorem 11 (Wiener's Tauberian theorem)

Let $g \in L^\infty(G)$. If $f \in L^1(G)$ is regular and

$$f * g(x) \to 0, \quad x \to \infty,$$

then we also have for every $h \in L^1(G)$ that

$$h * g(x) \to 0, \quad x \to \infty.$$

Wiener's Tauberian theory for operators

A function $g : \mathbb{R} \to \mathbb{C}$ is *slowly oscillating* if it is bounded and for every $\varepsilon > 0$ there exists some $R > 0$, $\delta > 0$ such that for $|x| > R$, $|y| < \delta$:

$$|g(x) - g(x - y)| < \varepsilon.$$

Essentially, this is some form of uniform continuity at infinity.

Theorem 12 (Pitt's refinement of Wiener's Tauberian theorem)

Let g be slowly oscillating and $f \in L^1(\mathbb{R})$ be regular. If

$$f * g(x) \to 0, \quad |x| \to \infty,$$

then we also have $g(x) \to 0$ as $|x| \to \infty$.

Wiener's Tauberian theory for operators

On lca groups, slowly oscillating functions are defined analogously: The condition $|x| > R$ is replaced by "x is outside of some fixed compact set" and "$|y| < \delta$" is replaced by "y is in some neighborhood of 0".
Our approach to Pitt's refinement for operators was through the following observation:

Theorem 13

$\mathrm{SO}(G) = \mathrm{BUC}(G) + B_0(G)$

Here,

$$B_0(G) = \{f \in L^\infty(G) : \ \forall \varepsilon > 0 \exists \ K \subset G \text{ cpt} : \ \|f \mathbf{1}_{K^c}\|_\infty < \varepsilon\}$$

$\mathrm{BUC}(G)$ can clearly be defined naturally in terms of the group action $G \ni x \mapsto \alpha_x(f)$:

$$\mathrm{BUC}(G) = \{f \in L^\infty(G) : \ \|\alpha_x(f) - f\|_\infty \to 0, \ x \to 0\}.$$

Another important observation was the following:

Lemma 9

Let $g \in L^\infty(G)$. Then, $g \in B_0(G)$ if and only if $\alpha_x(f) \to 0$ uniformly on compact subsets when $x \to \infty$.

We now return to operators: Motivated by the previous lemma, we set

$$B_0(\mathcal{H}) := \{B \in \mathcal{L}(\mathcal{H}) : \ \alpha_x(B) \to 0 \text{ in SOT}^* \text{ when } x \to \infty\}.$$

With this definition, we set:

$$\mathrm{SO}(\mathcal{H}) = \mathcal{C}(\mathcal{H}) + B_0(\mathcal{H}).$$

Wiener's Tauberian theory for operators

Indeed, one can show the following characterization of $\mathrm{SO}(\mathcal{H})$ which is analogous to the initial definition of a function:

Theorem 14 (RF, F. Luef, R. Werner '24)

Let $B \in \mathcal{L}(\mathcal{H})$. Then, $B \in \mathrm{SO}(\mathcal{H})$ if and only if the following property is satisfied for $A \in \{B, B^*\}$: For every $\varepsilon > 0$ there exists a nbhd O of 0 s.th. for each $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$ there exists a cpt set $K \subset \Xi$ with

$$\forall y \in O,\ z \in K^c: \ \|\alpha_z(A - \alpha_y(A))\varphi\| < \varepsilon.$$

At this stage, it is unclear if $B_0(\mathcal{H})$ is actually larger than $\mathcal{K}(\mathcal{H})$, or equivalently, if $\mathrm{SO}(\mathcal{H})$ is a larger class than $\mathcal{C}(\mathcal{H})$. We will return to this question in a moment. First, we answer our initial questions.

Wiener's Tauberian theory for operators

Theorem 15 (RF, F. Luef, R. Werner '24)

Let $B \in \mathcal{L}(\mathcal{H})$. Then, the following are equivalent:

1. $\alpha_x(B) = 0$ *for every* $x \in \partial\Xi$*;*
2. $A * B \in C_0(\Xi)$ *for every* $A \in \mathcal{T}^1(\mathcal{H})$*.*

Theorem 16 (RF, F. Luef, R. Werner '24)

Let $B \in \mathrm{SO}(\mathcal{H})$ *and* $A \in \mathcal{T}^1(\mathcal{H})$*. If*

$$A * B(x) \to 0, \quad x \to \infty,$$

then $B \in B_0(\mathcal{H})$*.*

Wiener's Tauberian theory for operators

What's the connection between all this and limit operator theory?
Behind the scenes (i.e., in the proofs and also in the underlying more general theory), we made the following definitions (again, we only present special cases):

$$\mathcal{A}_3(\Xi) := \{f \in L^\infty(\Xi) :\ \alpha_{z_\gamma}(f) \to \alpha_x(f) \text{ in c.o.-topology}\}$$
$$\mathcal{A}_2(\Xi) := \{f \in \mathcal{A}_3(\Xi) :\ \alpha_x(f) \in \mathrm{BUC}(\Xi) \text{ for } x \in \partial\Xi\}$$
$$\mathcal{A}_1(\Xi) := \{f \in \mathcal{A}_2(\Xi) :\ \{\alpha_x(f) :\ x \in \partial\Xi\} \text{ unif. equicont.}\},$$
$$\mathcal{A}_3(\mathcal{H}) := \{B \in \mathcal{L}(\mathcal{H}) :\ \alpha_{z_\gamma}(f) \to \alpha_x(f) \text{ in SOT}^*\},$$
$$\mathcal{A}_2(\mathcal{H}) := \{B \in \mathcal{A}_3(\mathcal{H}) :\ \alpha_x(B) \in \mathcal{C}(\mathcal{H}) \text{ for } x \in \partial\Xi\},$$
$$\mathcal{A}_1(\mathcal{H}) := \{B \in \mathcal{A}_1(\mathcal{H}) :\ \{\alpha_x(B) :\ x \in \partial\Xi\} \text{ unif. equicont.}\}.$$

Wiener's Tauberian theory for operators

We establish a correspondence theory between $\mathcal{A}_j(\Xi)$ and $\mathcal{A}_j(\mathcal{H})$, as well as between ideals of these algebras. Further, we noted that

$$\mathrm{SO}(\Xi) = \mathcal{A}_1(\Xi).$$

Based on this, we defined $\mathrm{SO}(\mathcal{H})$ as $\mathcal{A}_1(\mathcal{H})$. The properties of $\mathrm{SO}(\mathcal{H})$ mentioned earlier are consequences of this definition!

Once we had the characterization of $\mathrm{SO}(\mathcal{H})$ established, one problem was still left open: $\mathrm{SO}(\mathcal{H}) = \mathcal{C}(\mathcal{H})$ or $\mathrm{SO}(\mathcal{H}) \supsetneq \mathcal{C}(\mathcal{H})$?
Since $\mathrm{SO}(\mathcal{H}) = \mathcal{C}(\mathcal{H}) + B_0(\mathcal{H})$, this is equivalent to: $\mathcal{K}(\mathcal{H}) = B_0(\mathcal{H})$ or $\mathcal{K}(\mathcal{H}) \supsetneq B_0(\mathcal{H})$?
After several failed attempts to construct an operator in $B_0(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})$, we found an example in Halmos' *A Hilbert space problem book*.

Wiener's Tauberian theory for operators

From Halmos, *A Hilbert space problem book*:

> **Problem 177.** *If an operator A (on a Hilbert space of dimension $\aleph_0$) maps an orthonormal basis onto a sequence that converges strongly to 0, is* A *compact? What if* A *maps every orthonormal basis onto a strong null sequence?*

Wiener's Tauberian theory for operators

Solution to Problem 177 ("some"-part) and an operator in $B_0(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})$ for $\Xi = \mathbb{Z} \times \mathbb{T}$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ & & & & & & \ddots \end{pmatrix}.$$

Once this example is known, examples of similar nature can easily be constructed on $\mathcal{H} = L^2(\mathbb{R})$.

Thank you for your attention!