Analytic Continuation of Bergman spaces and commutative C^*-algebras

Gestur Ólafsson
Louisiana State University

Hannover, August 5–9, 2024

Partially joint work with:
Matthew Dawson and Raul Quiroga-Barranco
Khalid Bdarneh

Hilbert spaces of holomorphic functions

- In the simplest version quantization is about

Functions on a (symplectic) manifold. $\rightsquigarrow$ operators on a Hilbert space satisfying some "natural" axioms

- This process is often related to representation theory of topological groups (often Lie groups), Then the simplest way is the *integrated representations*:

$$\mathrm{L}^1(G) \to \mathrm{B}(\mathcal{H}), \quad \varphi \mapsto \pi(\varphi) = \int_G \varphi(x)\pi(x)dx$$

or $\mathcal{H} \otimes \overline{\mathcal{H}} =$ (different topologies) operators $\mathcal{H}$ or even $\mathrm{B}(\mathcal{H})$ with the action $a \cdot T = \pi(a)T\pi(a)^*$ (getting closer to QHA :))

- Another way is to use reproducing kernel Hilbert spaces $\mathcal{H} \subset \mathrm{L}^2(M,\mu)$ and a group G action eading to a unitary representation π on $\mathcal{H}$ and $\mathrm{L}^2(M,\mu)$. Then Toeplitz quantization:

$$\mathrm{L}^\infty(M) \ni \varphi \mapsto \mathrm{T}_\varphi \in \mathrm{B}(\mathcal{H}), \quad \mathrm{T}_\varphi F(x) = \int_M \varphi(m)F(m)K(x,m)d\mu(m)$$

- $K(x,m) = K_m(x)$ the reproducing kernel

$$\mathrm{pr}_{\mathcal{H}} : \mathrm{L}^2 \to \mathcal{H}, \quad \mathrm{pr}_{\mathcal{H}}(f)(m) = \int_M f(y)K(m,y)\,d\mu(y) = \langle f, K_m\rangle$$

L^2-needed?

Example

The Fock space on $\mathbb{C}^n$ with the action of the Heisenberg group or metaplectic group (or the Jacobi group) and the Bergman spaces on bounded symmetric domains with the action of coverings of the isometry group.

- Here the role of the L^2-space is to provide us with
  - a Hilbert space where the multiplication operator $\mathrm{M}_\varphi : f \mapsto \varphi f$ is well defined and bounded
  - The orthogonal projection $\mathrm{L}^2 \to \mathcal{H}$

Leading to: $\underbrace{\mathcal{H} \hookrightarrow L^2 \xrightarrow{\mathrm{M}_\varphi} L^2 \xrightarrow{\mathrm{pr}_\mathcal{H}} \mathcal{H}}_{\mathrm{T}_\varphi}, \ \varphi \in \mathrm{L}^\infty(M)$

- But there are examples of reproducing kernel Hilbert spaces with a irreducible unitary representation but
  - No L^2-space
  - Well defined Toeplitz quantization $\varphi \mapsto \mathrm{T}_\varphi$, for regular enough symbols: The analytic continuation of the Bergman spaces/holomorphic discrete series on bounded domains. Well know for 48 years, but not much used for Toeplitz operators so far.

Bergman spaces are associated to bounded domains: Classical bounded domains are

Type $(p \le q),\ n = p + q$

$$\mathrm{I}_{p,q}: \quad \{Z \in \mathrm{M}_{p,q}(\mathbb{C}) \mid \underbrace{\mathrm{I}_p - ZZ^*}_{\mathrm{K}(Z,Z)} > 0\}, \qquad \mathrm{I}_{1,n} = \mathrm{I}_{n,1} = \mathrm{B}^n$$

With action $\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}$

$$K_\lambda(Z, W) = \det(\mathrm{I}_p - ZW^*)^{-\lambda} = \det(\mathrm{I}_q - W^* Z)^{-\lambda}$$

$$\mathrm{II}_n: \quad \{Z \in \mathrm{M}_{n,n}(\mathbb{C}) \mid \underbrace{\mathrm{I}_n - ZZ^*}_{\mathrm{K}(Z,Z)} > 0, Z^\top = Z\} = \mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n)$$

$$\mathrm{III}_n: \quad \{Z \in \mathrm{M}_{n,n}(\mathbb{C}) \mid \underbrace{\mathrm{I}_n - ZZ^*}_{\mathrm{K}(Z,Z)} > 0, Z^\top = -Z\} = \mathrm{SO}^*(2n)/K$$

$$\mathrm{IV}_n: \quad \{z \in \mathbb{C}^n \mid \underbrace{|z^\top z|^2 + 1 - 2z^* z}_{\mathrm{K}(Z,Z)} > 0,\ |z^\top z| < 1\} = \mathrm{SO}(2, n)/\mathrm{SO}(n) \times \mathrm{SO}(2)$$

(the Lie ball)

The disk

- Note that $\mathbb{D} = \mathrm{B}^1$ fits into the series

$$\mathrm{I}_{1,n} = \mathrm{B}^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\} \quad \text{the unit ball in } \mathbb{C}^n \quad \text{part of } \mathrm{I}_{p,q}$$

- But, as $\mathbb{D} \simeq \mathbb{C}^+ = \mathbb{R} + i\mathbb{R}^+$ is a tube type domain, we can also view it as part

$$\begin{aligned}
\text{The real case} &: \quad \mathrm{II}_n \simeq \mathrm{Sym}_n(\mathbb{R}) + i\mathrm{Sym}_n^+(\mathbb{R}) \\
\text{The complex case} &: \quad \mathrm{I}_{n,n} \simeq \mathrm{Sym}_n(\mathbb{C}) + i\mathrm{Sym}_n^+(\mathbb{C}) \\
&\simeq \mathrm{SU}(n,n)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n)) \\
\text{Lie ball} &: \quad \mathrm{IV}_n \simeq \mathbb{R}^n + i\left\{ y \in \mathbb{R}^n \,\middle|\, \begin{array}{c} y_n^2 - (y_1^2 + \cdots + y_{n-1}^2) > 0 \\ y_n > 0 \end{array} \right\}.
\end{aligned}$$

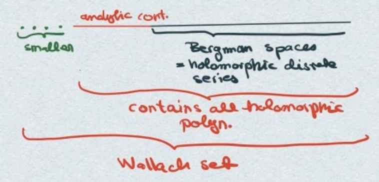
$\mathrm{Sym}_n(\mathbb{F}) = \{Z \in \mathrm{M}_n(\mathbb{F}) \mid Z^* = Z\}$ and $_+$ stands for positive definite.

Bergman spaces on bounded domains $\mathbb{D} = G/K$ and their analytic continuation

- Bergman spaces $\mathbb{A}^2_\lambda(\mathbb{D}) = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D}, \mu_\lambda)$ with μ_λ a probability measure. Reproducing kernel Hilbert space of holomorphic functions. Furthermore, there exists a known polynomial $k(z,w)$ ($= \det K(Z,W)$ for the classical domains), such that $k(z,z) > 0$ on $\mathbb{D}$ and

$$K^\lambda(z,w) = K^\lambda_w(z) = k(z,w)^{a\lambda+b}, \quad a, b \text{ known numbers.}$$

- M. Vergne & H. Rossi, Acta Math 1976, H.P. Jakobsen 1983, Wallach 1985, Ørsted: $\mathrm{SU}(n,n)$, Faraut-Koranyi, ... : There is a bigger set where the reproducing kernel is positive definite, and hence defining a reproducing kernel Hilbert space of holomorphic functions.



Ideas

- The kernel $K^\lambda(z, w)$ is defined for all λ. The question is: For which λ is $K^\lambda(z, w)$ positive definite.

1) The original idea by Vergne + Rossi: Use the unbounded realization. Write the reproducing kernel as a Laplace/Fourier transform and use that to determine the set where K^λ is positive definite.
2) Analytic continuation of the inner product of homogeneous polynomials and determine the set where the inner product is non-negative.
3) Bernstein polynomial: There exists a differential operator D, a polynomial $b(\lambda)$ and a natural number r_0 such that

$$DK_\lambda(z, z) = b(\lambda)K_{\lambda - r_0}(z, z)$$

Good for analytic continuation of Toeplitz operators.

- I will discuss (2) and (3) for the ball.

Bergman spaces on the ball, parametrized according to rep theory tradition, $\lambda = \alpha + n + 1$

- For $\lambda > n$ let

$$\mu_\lambda(dz) = \frac{\Gamma(\lambda)}{n!\Gamma(\lambda - n)}(1 - \|z\|^2)^{\lambda - n - 1} dz = c_\lambda (1 - \|z\|^2)^{\lambda - n - 1} dz$$

a probability measure on the Ball $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$. Note

$$(1 - \|z\|^2)^{-n-1} dz \text{ is } G - invariant.$$

- Bergman space: $\mathrm{A}^2_\lambda(\mathbb{B}^n) = \mathrm{L}^2(\mathbb{B}^n, \mu_\lambda) \cap \mathcal{O}(\mathbb{B}^n)$,
- With $p_\beta(z) = z_1^{\beta_1} \cdots z_n^{\beta_1} \in \mathrm{A}^2_\lambda(\mathbb{B}^n)$, $\beta \in \mathbb{N}_0$, we have

$$\langle p_\beta, p_\gamma \rangle_\lambda = \delta_{\beta,\gamma} \frac{\beta! \Gamma(\lambda)}{\Gamma(\lambda + |\beta|)} (= w_\lambda(\beta)) > 0), \quad \lambda > 0$$

Hence for $F(z) = \sum_{\beta \in \mathbb{N}_0^n} f_\beta p_\beta(z)$, $\quad G(z) = \sum_{\beta \in \mathbb{N}_0^n} g_\beta p_\beta(z) \in \mathcal{O}(\mathbb{B}^n)$ we can define an inner product

$$\langle F, G \rangle_\lambda := \sum_\beta f_\beta \overline{g_\beta} w_\lambda(\beta)$$

Method (2): Realization as sequence space

- Leading to a Hilbert space $A_\lambda(\mathbb{B}^n) \subset \mathcal{O}(\mathbb{B}^n)$, $\lambda > 0$, isomorphic to the sequence space

$$\ell^2(\mathbb{N}_0^n, w_\lambda) = \{(a_\beta) \mid \sum_\beta |a_\beta|^2 w_\lambda(\beta) < \infty\}$$

and

1) $\{\varphi_{\lambda,\beta} = c_\lambda(\beta) p_\beta\}_\beta$ orthonormal basis (onb) for $\mathrm{A}^2_\lambda(\mathbb{B}^n)$.
2) Reproducing kernel $K^\lambda(z, w) = (1 - \langle z, w \rangle)^{-\lambda}$.

But: For $0 < \lambda \leq n$ there is no $\mathrm{L}^2(\mu_\lambda)$ and hence no projection.

Idea Yan 2000/for the Ball: Chailuek and Hall, 2009

- Let $\mathrm{E} = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ the Euler operator and let $\overline{\mathrm{E}} = \sum \overline{z}_j \frac{\partial}{\partial \overline{z}_j}$.
- $p(z) \in \mathrm{P}_m(\mathbb{C}^n)$ homogeneous polynomial of degree $m \Rightarrow \mathrm{E}p = mp$.
- In particular $Ep_\beta = |\beta| p_\beta$ and

$$(\lambda \mathrm{I} + E)\, p_\beta = (\lambda + |\beta|) p_\beta$$

which implies that (shift in the parameter!)

$$\langle p_\gamma, \left(\mathrm{I} + \frac{1}{\lambda} E\right) p_\beta \rangle_{\lambda+1} = \langle \left(\mathrm{I} + \frac{1}{\lambda} E\right) p_\gamma, p_\beta \rangle_{\lambda+1} = \langle p_\gamma, p_\beta \rangle_\lambda.$$

Lemma (CH09)

We have for $f, g \in \mathrm{L}^2(\mu_\lambda) \cap C_b^1(\mathbb{B}^n)$ ($\leftarrow$ continuously differentiable with bounded derivatives) and $E_\lambda^\pm = \mathrm{I} \pm \frac{1}{\lambda}\mathrm{E}$ and similarly $\overline{\mathrm{E}}_\lambda$.

1) $\mathrm{E}_{\lambda+1}^-(1 - |z|^2)^{\lambda+1} = \overline{\mathrm{E}}_{\lambda+1}^-(1 - |z|^2)^{\lambda+1} = (1 - |z|^2)^\lambda$.
2) *If $\lambda > n$ then $\langle f, g \rangle_{\mathrm{L}^2(\mu_\lambda)} = \langle f, E_\lambda^+ g \rangle_{\mathrm{L}^2(\mu_{\lambda+1})} = \langle E_\lambda^+ f, g \rangle_{\mathrm{L}^2(\mu_{\lambda+1})}$.*

The differential shift operator

- Define differential operators $\mathrm{A}_m(\lambda) = \prod_{j=m}^{2m-1} \mathrm{E}^+_{\lambda+j}$ and $\mathrm{B}_m(\lambda) = \prod_{j=0}^{m-1} \mathrm{E}^+_{\lambda+j}$ and note that

$$\mathrm{A}_m(\lambda)\mathrm{B}_m(\lambda) = \prod_{j=0}^{2m-1} \left(\mathrm{I} + \frac{1}{\lambda + j} E \right).$$

Theorem (CH 09)

The following holds for $m \in \mathbb{N}$ and $\lambda > 0$

(0) *If $F \in \mathcal{O}(\mathbb{B}^n)$ then $F \in \mathrm{A}^2_\lambda(\mathbb{B}^n) \Leftrightarrow \mathrm{E}^j f \in \mathrm{A}^2_{\lambda+2m},\ \forall 0 \leq j \leq m$.*

(1) *If $F, G \in \mathrm{A}^2_\lambda(\mathbb{B}^n)$ then $\langle F, G \rangle_\lambda = \langle A_m(\lambda) F, B_m(\lambda) G \rangle_{\lambda+2m}$.*

- (1) can be used to give a new characterization of the analytic continuation of the Bergman spaces and the analytic continuation of the Toeplitz operators.

Analytic continuation of Toeplitz operators, I

- Start with $\lambda > n$ and $\mathrm{E}_\lambda^-(1-|z|^2)^{\lambda+1} = (1-|z|^2)^\lambda$
- $\varphi \in C_b^1(\mathbb{B}^n) \Rightarrow$:

$$\begin{aligned}\mathrm{T}_\varphi^\lambda F(z) &= c_{\lambda+1}\int \varphi(w)F(w)K(z,w)\mathrm{E}_\lambda^-(1-|w|^2)^{\lambda+1}dw \\ &= c_{\lambda+1}\int \mathrm{E}_\lambda^{-,\vee}(\varphi F\bar{K}_z)(w)(1-|w|^2)^{\lambda+1}dw \quad \leftarrow \text{ partial integration}\end{aligned}$$

- Iterating with $\mathrm{D}_m(\lambda) = c_{\lambda,m}\mathrm{E}_\lambda^{-,\vee}\cdots\mathrm{E}_{\lambda+2m-1}^{-,\vee}$ we get:

$$\mathrm{T}_\varphi^\lambda F(z) = \int \mathrm{D}_m(\lambda)(\varphi F\bar{K}_z)(w)(1-|w|^2)^{\lambda+2m}dw. \tag{1}$$

with the coefficients of $D_m(\lambda)$ holomorphic on $\mathbb{C}\setminus -\mathbb{N}_0$.

Analytic continuation of Toeplitz operators, II

Theorem

For $\lambda > 0$ let $m \in \mathbb{N}$ be so that $\lambda + 2m > n$. Denote by $C_b^{2m}(\mathbb{B}^n)$ the space of the of functions on $\mathbb{B}^n$ such that

- *all derivative $\mathrm{D}^\alpha \varphi$, $|\alpha| \leq 2m$, exists*
- *and are bounded on $\mathbb{B}^n$.*

Then (1) is analytic in λ and defines an analytic continuation of $\lambda \mapsto \mathrm{T}_\varphi^\lambda \in \mathrm{B}(\mathcal{A}_\lambda^2)$ on $(0, \infty)$.

Remark

There is no reason to expect that the norm of $\mathrm{T}_\varphi^\lambda$ is bounded by $\|\varphi\|_\infty$ ss the definition uses derivatives of the symbol. In fact it was shown in [CH09] that with $\varphi(z) = |z|^2$. Then there is no constant $C > 0$ such that

$$\|\mathrm{T}_\varphi^\lambda\| \leq C\|\varphi\|_\infty.$$

The group $\mathrm{SU}(n,1) \simeq \mathrm{SU}(1,n)$ acting on $\mathbb{B}^n$ and the representation

- Let $G = \mathrm{SU}(n,1)$. Write r elements of G as

$$g = g(A; d; u, v) = \begin{pmatrix} A & u \\ v^* & d \end{pmatrix}, \quad A \in \mathrm{M}_n(\mathbb{C}) \text{ and } \det g = 1, u, v \in \mathbb{C}^n$$

- The action of G on $\mathbb{B}^n$ is $g \cdot z = (v \cdot z + d)^{-1} (Az + u)$ and

$$\mathbb{B}^n \simeq G/K, \quad K = \mathrm{U}(n) \hookrightarrow \mathrm{SU}(n,1),\ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1/\det A \end{pmatrix}.$$

Theorem

Let $j_\lambda(g,z) = (v \cdot z + d)^\lambda$ (if λ is not an integer, then coverings or projective rep of G). For $\lambda > 0$ define

$$\pi_\lambda(g)F(z) = j_\lambda(g^{-1}, z)F(g^{-1}z), \quad F \in \mathbb{A}_\lambda(\mathbb{B}^n).$$

Then

- $j_\lambda(g,z)\overline{j_\lambda(g,w)}K^\lambda(g \cdot z, g \cdot w) = K^\lambda(z,w)$.
- π_λ *defines an irreducible unitary representation of G acting on* $\mathrm{A}^2_\lambda(\mathbb{B}^n)$.

Connection to rep Theory, Intertwining operators and invariant symbols

- For a measurable function $\varphi : \mathbb{B}^n \to \mathbb{C}$ and $g \in G$ let $g \cdot \varphi = \ell_g \varphi = \varphi \circ \ell_{g^{-1}}$.

Definition

$H \subset G$ a closed subgroup then φ is H-invariant if for all $h \in H : h \cdot \varphi = \varphi$.

- Denote by $\mathcal{T}_\lambda^H \subset \mathrm{B}(\mathcal{A}_\lambda^2)$ the C^*-algebra generated by all H-invariant symbols (with needed smoothness). It is well known that for $\lambda > n$
  - $\mathcal{T}_\lambda^K$ is commutative
  - H maximal abelian $\Rightarrow \mathcal{T}_\lambda^H$ is commutative (Q-B+V, 2007/2008).

- If $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ are unitary representations of a group H, then $T \in \mathrm{B}(\mathcal{H}, \mathcal{K})$ is an intertwining operator if $\mathrm{T}\pi(a) = \rho(a)\mathrm{T}$, $a \in H$.

- $\mathcal{I}_H(\pi, \rho) =$ the space of intertwining operators.

- If $\pi = \rho$ then $\mathcal{I}(\pi) = \mathcal{I}(\pi, \pi)$ is a von Neumann algebra.

Lemma

Let $\lambda > 0$ and $a \in G$ and $\varphi \in C_b^{2m}(\mathbb{B}^n)$. Then

$$\mathrm{T}_\varphi^\lambda \pi(a) = \pi(a) T_{\ell_{a^{-1}}\varphi}^\lambda .$$

The proof

Proof.

This is well known (see as an example D-Q-B-Ó 2015) for $\lambda > n$. For the general case let $F \in \mathrm{A}^2_\lambda(\mathbb{B}^n)$, $z \in \mathbb{B}^n$ then $\lambda > n$

$$(\mathrm{T}^\lambda_\varphi \pi(a)F)(z) - (\pi(a)\mathrm{T}^\lambda_{\ell_{a^{-1}}\varphi}F)(z) = 0$$

The left hand side is analytic in λ, hence $= 0$ for all $\lambda > 0$. □

Theorem (Generalized Englis Theorem, Bdarneh 2023)

Let H be a compact subgroup of $\mathrm{SU}(n,1)$. For every $\lambda > 0$, the space $\mathcal{T}^\lambda_H$ is dense in $\mathcal{I}(\pi_\lambda|_H)$ in the weak and strong operator topology.

- All of this leads to generalization of D–Ó–Q-B, 2015:

Theorem

Let $\lambda > 0$. Let $H \subset G$ be a closed subgroup and let $C^{2m}_b(\mathbb{B}^n)^H$ be the space of H-invariant symbols. Denote by $\mathcal{T}^\lambda_H$ the C^*-algebra generated by $\{\mathrm{T}^\lambda_\varphi \mid \varphi \in C^{2m}_b(\mathbb{B}^n)\}$. Then the following holds:

- *If $\mathcal{I}(\pi_\lambda|_H)$ is abelian, then the algebra $\mathcal{T}^\lambda_H$ is abelian.*
- *If H is compact, then $\mathcal{T}^\lambda_H$ abelian $\Leftrightarrow \mathcal{I}(\pi|_H)$ abelian.*

How to get the spectrum: Restriction Injectivity

- Let $H \subset \widetilde{G}$ be a closed subgroup and assume that there exists a point $z_0 \in \mathbb{B}^n$ such that $M = H \cdot z_0$ is restriction injective: If $F \in \mathcal{O}(\mathbb{B}^n)$ and $F|_M = 0$ then $F = 0$.

- Let $D_\lambda = j_\lambda(\cdot, z_0)$ and let

$$\chi_\lambda = j_\lambda(\cdot, z_0)^{-1}|_{\widetilde{G}^{z_0}} \quad \text{a homomorphism.}$$

- Defines a line bundle $\mathcal{V}_\lambda \to M$ and $D_\lambda \in \Gamma(\mathcal{V}_\lambda)$ (= space of sections). We assume that for the moment that, with $L = G^{z_0}$:

$$D_\lambda \in \mathrm{L}^2(\mathcal{V}_\lambda) = \left\{ f : H \to \mathbb{C} \;\middle|\; \begin{matrix} f(hs) = \chi_\lambda(s)^{-1} f(h) \\ h \in H, s \in L \end{matrix}, \int_{H/L} |f(h)|^2 d\mu_M(h) < \infty \right\}.$$

- Define $R_\lambda : \mathrm{A}^2_\lambda(\mathbb{B}^n) \to \mathrm{L}^2(\mathcal{V}_\lambda)$ by

$$R_\lambda F(h) = D_\lambda(h) F(h \cdot z_0).$$

Densely defined and closed. Polar decomposition

$$R^*_\lambda : \mathrm{B}^2_\lambda(M) := \overline{\mathrm{Im} R_\lambda} \to \mathrm{A}^2_\lambda(\mathbb{B}^n), \quad R^*_\lambda = U_\lambda |R^*_\lambda|.$$

$R_\lambda R_\lambda^*$

- All the operators commute with the action of H and

$$R_\lambda^* f(z) = \langle R^* f, K_z^\lambda \rangle = \langle f, R_\lambda K_z \rangle = \int_M f(h)\overline{D_\lambda(h)} K(z, hz_0) d\mu_M(h)$$

Hence

$$R_\lambda R_\lambda^* f(h) = \int_M f(x) D_\lambda(x) \overline{D_\lambda(h)} K(z, hz_0) d\mu_M(h)$$
$$= \int_M f(x) j_\lambda(h^{-1}x, z_0) K^\lambda(h^{-1}x \cdot z_0, z_0) d\mu_M(h).$$

Theorem (DÓQ-B)

Let $H \subset G$ closed subgroup and $\varphi \in C_b^{2m}(\mathbb{B}^n)^H$. Under the assumption that $D_\lambda \in L^2(\mathcal{V}_\lambda)$ we have that:

1) $U_\lambda^* \mathrm{T}_\varphi^\lambda U_\lambda = |R_\lambda|^{-1}(R_\lambda \mathrm{T}_\varphi R_\lambda^*)$.
2) $R_\lambda \mathrm{T}_\varphi R^* f = f * \Phi_{\lambda,\varphi}, \quad \Phi_{\lambda,\varphi}(h) = \Phi_{\lambda,\varphi}^H(h) = D_\lambda(h)\langle \varphi K_{z_0}^\lambda, K_{h\cdot z_0}^\lambda \rangle_\lambda$.
3) $R_\lambda R_\lambda^* f = f * \Phi_\lambda$ *where* $\Phi_\lambda(h) = \Phi_\lambda^H(h) = D_\lambda(h) K_{z_0}^\lambda(h \cdot z_0)$.

- Note: We do not need the form of the inner product in $\mathrm{A}_\lambda^2(\mathbb{B}^n)$ except in (3) and here we can work out the spectrum for big parameters and then use analytic continuation.

Examples: The maximal torus

- There are five classes of maximal abelian subgroup in $\mathrm{SU}(n,1)$:
  - Maximal torus:
    $T_n = \{\mathrm{diag}(at_1, \dots, at_n, a) \mid a, t_j \in \mathbb{T}, a^{n+1} t_1 \cdots t_n = 1\} \simeq \mathbb{T}^n$.
  - Quasi-elliptic:
    $$E_n = \left\{ k(t,a) = \begin{pmatrix} \mathrm{diag}(at) & \\ & a \end{pmatrix} \,\middle|\, t \in \mathbb{T}^n, a \in T, \det k(t,a) = 1 \right\}$$
  - Quasi-parabolic:
    $$P_n = \left\{ p(t,a,y) = \begin{pmatrix} \mathrm{diag}(at) & & \\ & a & ay \\ & & a \end{pmatrix} \,\middle|\, \begin{array}{c} t \in \mathbb{T}^{n-1}, a \in T, \\ \det p(t,a,y) = 1 \end{array} \right\}$$
  - Quasi-hyperbolic:
    $$H_n = \left\{ h_{t,a,r} = \begin{pmatrix} \mathrm{diag}(at) & & \\ & ar & \\ & & a/r \end{pmatrix} \,\middle|\, \begin{array}{c} t \in \mathbb{T}^{n-1}, a \in T, \\ r \in \mathbb{R}^+, \det h(t,a,r) = 1 \end{array} \right\}$$
  - Also nilpotent N_n and quasi-nilpotent QN_n.

- As mentioned before: Q-B/V: If $H = T_n, E_n, P_n, H_n \Rightarrow \mathcal{T}^H$ is commutative.

The maximal torus

The simplest example is the maximal torus

$$T = \{\mathrm{diag}(at_1, \ldots, at_n, a) \mid a, t_j \in \mathbb{T}, a^{n+1} t_1 \cdots t_n = 1\} \simeq \mathbb{T}^n.$$

The corresponding subgroup in $\widetilde{G}$ is

$$\widetilde{\mathbb{T}^n} = \{v = (t_1, \ldots, t_n, x) \in \mathbb{T}^n \times \mathbb{R} \mid e^{2\pi(n+1)ix} t_1 \cdots t_n = 1\}.$$

- φ $\mathbb{T}^n$-invariant $\Rightarrow$ the monomials φ_β are eigenvectors for T_φ and

$$T^\lambda_\varphi(F) = \sum_\beta \langle T^\lambda_\varphi \varphi_\beta, \varphi_\beta \rangle_\lambda \langle F, \varphi_\beta \rangle_\lambda \varphi_\beta\,, \quad \lambda > 0.$$

- Taking $z_0 = \frac{1}{\sqrt{2n}}(1, \ldots, 1)$ we get

$$D_\lambda(v) = e^{-2\pi i \lambda x} \quad \text{and} \quad \Phi^H_\lambda(v) = e^{-2\pi \lambda x}(1 - \frac{1}{2n}\sum t_j)^{-\lambda}.$$

Example 1: The maximal torus

- As $|D_\lambda| \in \mathrm{L}^2(\mathbb{T}^n)$ for all $\lambda > 0$ it follows that the methods from [DÓQ-B] works for all $\lambda > 0$. In particular

1) $\pi_\lambda|_H(t,x)f(s) = e^{2\pi i x\lambda} f(t^{-1}x)$, $t, s \in \mathbb{T}^n$, $x \in \mathbb{R}$, acting on
2) $\mathrm{Im} U_\lambda \overline{\{\sum_\beta a_\beta s^\beta \mid a \in \mathbb{C}\}} \subset \mathrm{L}^2(\mathbb{T}^n) \simeq \mathcal{A}^2_\lambda$, the Hardy space and the Toeplitz operators act by as convolution operators with $D_\lambda(h)\langle \varphi K^\lambda_{z_0}, K^\lambda_{hz_0}\rangle_\lambda$. Take Fourier transform $\rightsquigarrow$ acts as multiplication operators.

Example: The Quasi-Parabolic case

- In this case

$$H = \left\{ \operatorname{diag}\left(at, \begin{pmatrix} a(1+\frac{iy}{2}) & \frac{aiy}{2} \\ \frac{-iay}{2} & a(1-i\frac{y}{2}) \end{pmatrix} \right) \,\middle|\, \begin{matrix} t \in \mathbb{T}^{n-1} \\ a^{n+1} t_1 \cdots t_{n-1} = 1 \\ y \in \mathbb{R} \end{matrix} \right\}.$$

- The orbit through $z_0 = \frac{1}{\sqrt{2(n-1)}}(1, \ldots, 1, 0)$ is

$$M = \left\{ \left(\frac{2}{2-iy} \frac{t}{\sqrt{2(n-1)}}, \frac{iy}{2-iy} \right) \,\middle|\, t \in \mathbb{T}^{n-1}, y \in \mathbb{R} \right\}$$

- $D_\lambda(q) = j_\lambda(q, z_0) = 2^\lambda e^{-2\pi i \lambda x}(2-iy)^{-\lambda} \in L^2(M)$ iff $\lambda > 1/2$.
- The stabilizer is $\mathbb{R}$ and

$$\mathrm{L}^2(\mathcal{V}_\lambda) = L^2(M) \simeq \mathrm{L}^2(\mathbb{T}^{n-1} \times \mathbb{R})$$

with the natural action of t and y and $x \in \mathbb{R}$ acting by $e^{2\pi i \lambda x}$.

Definition of R

- $D_y = \frac{1}{i}\frac{\partial}{\partial y}$ is densely defined, closed and injective on $L^2(M)$ and

$$D_y(1-iy)^{-\lambda} = \lambda(1-iy)^{-\lambda-1} \in L^2 \quad \text{if } \lambda > -1/2.$$

- Define: $\widetilde{R}_\lambda F = D_y R_\lambda F = \frac{1}{i}\frac{\partial}{\partial y} D_\lambda F|_M$.

- R_λ interchange $\pi_\lambda|_H$ and the left regular representation. D_y commutes with translation and hence $\widetilde{R}_\lambda$ is an intertwining operator, giving us a way to combine with the Fourier transform to realize T_φ, φ H-invariant, as a multiplication operator. The spectral function for $\mathrm{T}_\varphi^\lambda$ is analytic and hence it is enough to determine it for big λ.

- For all other cases one can use constant coefficient differential operators or power of the Laplacian Δ to construct restriction operators to diagonalize Toeplitz operators with invariant symbols. Everything is explicit and

  - Quasi-Hyperbolic: $\lambda > 0$,
  - Nilpotent: $\lambda > (n+1)/2$.
  - Quasi-Nilpotent: $\lambda > \frac{n+1-k}{4}$ where k is the dimension of a maximal torus in H.