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Norwegian University of Science and Technology

Quantum Time–Frequency Analysis

Workshop on Quantum Harmonic Analysis in Hannover

Based on joint work with Franz Luef

August 5, 2024

Notation

We use the following notation:

$$
\begin{aligned}
\pi(z)f &= M_\omega T_x f \\
V_g f(z) &= \langle f, \pi(z)g \rangle \\
\alpha_z(S) &= \pi(z) S \pi(z)^* \\
Pf(t) &= f(-t) \\
S \star T &= \operatorname{tr}(S\alpha_z(\check{T})), \\
f \star S &= \int_{\mathbb{R}^{2d}} f(z)\alpha_z(S)\, dz
\end{aligned}
$$

Notation

$$W(f,g)(z) = \int_{\mathbb{R}^d} f(x+\tfrac{t}{2})\overline{g(x-\tfrac{t}{2})} \cdot e^{-2\pi i\omega t}\,dt$$

$$\mathcal{F}_\Omega f(z) = \int_{\mathbb{R}^{2d}} f(z')e^{-2\pi i(x'\omega - x\omega')}$$

$$\mathcal{F}_W(S)(z) = e^{-i\pi x\cdot\omega} tr(\pi(-z)S)$$

$$\mathcal{F}_\Omega(W(f,g))(z) = e^{\pi i x\omega} V_f g(z)$$

$$\sigma_{f\otimes g} = W(f,g)$$

$$M^{p,q}(\mathbb{R}^d) = \{f \in \mathscr{S}' : V_{\varphi_0} f \in L^{p,q}(\mathbb{R}^{2d})\}$$

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A Projective Representation on Hilbert-Schmidt Operators

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QHA considers unitary representation α_z, corresponding to translations of the Weyl symbol. Decomposition results have thus considered g-frames due to limitations discretising translations.

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$$\gamma_{w,z}(S) := \pi(z)S\pi(w)^*.$$

For rank one $S = f \otimes g$,

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For rank one $S = f \otimes g$,

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Corresponds to translation and modulation of Weyl symbol:

$$\sigma_{\gamma_{w,z}S} = c_{w,z}\pi(U(w,z)\sigma_S$$

where

$$U = \left(\frac{w_1+z_1}{2}, \frac{w_2+z_2}{2}, w_2 - z_2, z_1 - w_1\right).$$

What is a Modulation for Operators?

Recalling relation between Weyl symbol and Fourier-Wigner;

$$\mathcal{F}_W(\beta_w(S)) = T_w \mathcal{F}_W(S)$$

Modulation of Weyl symbol is given by

$$\beta_w(S) := e^{-i\pi w_1 \cdot w_2/2} \pi(\tfrac{w}{2}) S \pi(\tfrac{w}{2}) = \pi(\tfrac{w}{2}) S \pi(-\tfrac{w}{2})^*.$$

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Parity operator P plays analogous role to identity operator I in translation case:

$$\alpha_z(I) = I, \quad \beta_w(I) = e^{-\pi i w_1 w_z} \pi(w) \cdot I$$
$$\beta_w(P) = P, \quad \alpha_{z/2}(P) = e^{-\pi i z_1 z_2} \pi(z) \cdot P$$

What is a Modulation for Operators? (cont.)

Given some $S \in \mathcal{M}^\infty$:

$$S = \int_{\mathbb{R}^{2d}} e^{i\pi x\cdot\omega} \mathcal{F}_W(S)\pi(z)\, dz = \int_{\mathbb{R}^{2d}} e^{-i\pi x\cdot\omega} \sigma_S(\tfrac{z}{2})\pi(z) P\, dz.$$

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Just as I is only translation invariant operator, so is P the only modulation invariant operator. However, for a lattice $\Lambda \subset \mathbb{R}^{2d}$, we can consider the space of Λ-modulation invariant operators:

$$\mathcal{L}_\Lambda := \{T \in \mathcal{L}(L^2) :\ \beta_\lambda(T) = T\, \forall \lambda \in \Lambda\}.$$

Just as Λ-translation invariant operators correspond to generalised Gabor multipliers, Λ-modulation invariant operators correspond to Fourier-Wigner periodisations of $\mathcal{M}^1$ operators, and as such have discretisation in terms of shifted parity operators.

What is a Modulation for Operators? (cont.)

Λ-modulation invariant operators are then reconstructable from a discretised convolution with appropriate S;

$$T = \frac{1}{\sigma_S(0)} \sum_{\Lambda^\circ} c_{\lambda^\circ} T \star S(\lambda^\circ)\, \pi(2\lambda^\circ) P.$$

We have also a Janssen type discrete representation of T in terms of shifted parity operators, and the symbol calculus

$$\mathcal{F}_W(S)\mathcal{F}_W(T) = \mathcal{F}_\Omega \mathcal{F}_W(ST)$$

for appropriate Λ.

Matrix Coefficients and Integrated Representation

Given our representation $\gamma_{w,z} : S \mapsto \pi(z)S\pi(w)^*$, can consider the matrix coefficients and integrated representation:

$$Q_S T(w,z) := \langle T, \gamma_{w,z}(S)\rangle_{\mathcal{HS}}$$

$$Q_S^* F := \int_{\mathbb{R}^{4d}} F(w,z)\gamma_{w,z}(S)dz\,dw$$

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Have then reconstruction formula, if $S = \sum_n f_n \otimes g_n$ and $T = \sum_n \psi_n \otimes \phi_n$:

$$\begin{aligned} Q_S^* Q_S T &= \int_{\mathbb{R}^{4d}} \sum_{n,m} V_{f_n}\psi_m(z) \cdot \overline{V_{g_n}\phi_m(w)} \cdot \gamma_{w,z}(S)\, dz\, dw \\ &= \sum_{i,n,m} \left(\int_{\mathbb{R}^{2d}} V_{f_n}\psi_m(z)\pi(z) f_i\, dz \right) \otimes \left(\int_{\mathbb{R}^{2d}} V_{g_n}\phi_m(w)\pi(w) g_i\, dw \right) \\ &= \|S\|_{\mathcal{HS}} \cdot T \end{aligned}$$

The Rank–One cases

If $S = f \otimes g$, then $Q_S T$ is the (cont.) Gabor matrix [4] [2] [1]

$$Q_S T = \langle T\pi(w)g, \pi(z)f\rangle_{L^2}.$$

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Given mask $F \in L^2(\mathbb{R}^{4d})$,

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is the "bilocalisation operator" [3].The reproducing property is used in the rank one case in [5] to show decay properties for Schwartz operators.

Extending to $\mathcal{M}^{p,q}$ Spaces

Considering now dual pairing $(\mathfrak{S}, \mathfrak{S}')$, with associated dual action of projective tensor product space given by the generalised trace. Then define the polarised Cohen's class as

$$Q_S T(w,z) = \langle T, \gamma_{w,z}(S) \rangle_{\mathfrak{S}',\mathfrak{S}}.$$

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Recall that $\sigma_{\gamma_{w,z}(S)} = c_{w,z}\pi(U(w,z))\sigma_S$, where U is unitary. Thus, starting with the $p = q = 1$ case, the space $\mathcal{M}^1$ is given by

$$\mathcal{M}^1 := \{S \in \mathfrak{S}' : Q_S S \in L^1(\mathbb{R}^{4d})\}.$$

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$\mathcal{M}^{p,q}$ spaces are then given by

$$\mathcal{M}^{p,q} := \{T \in \mathfrak{S}' : Q_S T \in L^{p,q}(\mathbb{R}^{4d})\}$$

for any $S \in \mathcal{M}^1$, and satisfy the coorbit correspondence property

$$F = Q_S T \iff F \circledast Q_S S = F$$

where $\circledast$ is the tensorisation of the twisted convolution, and $S \in \mathcal{M}^1$ with $\|S\|_{\mathcal{HS}} = 1$.

Gabor Frames for Operators

We say that S generates a Gabor frame for $\mathcal{HS}$ on $\Lambda \times M$ if the set

$$\{\gamma_{\lambda,\mu}(S)\}_{(\lambda,\mu)\in\Lambda\times M}$$

is a frame for $\mathcal{HS}$. Have for example, if f, g generate Gabor frames on M, Λ respectively, then $f \otimes g$ generates operator Gabor frame on $\Lambda \times M$.

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$$T \in \mathcal{M}^{p,q} \iff \{Q_S T\}_{\Lambda\times M} \in \ell^{p,q}(\Lambda \times M)$$

for an $S \in \mathcal{M}^1$ which generates a Gabor frame on $\Lambda \times M$.

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for an $S \in \mathcal{M}^1$ which generates a Gabor frame on $\Lambda \times M$.Given some $T \in \mathcal{M}^{p,q}$, we have a decomposition

$$T = \sum_{n\in\mathbb{N}} s_n \phi_n \otimes \psi_n,$$

where $\{s_n\}_N \in \ell^q$, $\phi_n \in M^1(\mathbb{R}^d)$ and $\psi_n \in M^p(\mathbb{R}^d)$ normalised.

Properties of $\mathcal{M}^{p,q}$ Spaces

Since U is unitary, $\mathcal{M}^p$ corresponds to operators with symbols in $M^p(\mathbb{R}^{2d})$.
Generally, $M^{p,q}(\mathbb{R}^{2d})$ do *not* correspond to symbols in a modulation space, due to U. Can consider the spaces $\mathcal{M}^{p,q}$ as $M^p(\mathbb{R}^d)$–valued modulation spaces in the sense of [6].

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$$\begin{aligned}
\mathcal{M}^1 &= \mathcal{N}(M^\infty(\mathbb{R}^d); M^1(\mathbb{R}^d)) \\
\mathcal{M}^2 &= \mathcal{HS} \\
\mathcal{M}^\infty &= \mathcal{L}(M^1(\mathbb{R}^d); M^\infty(\mathbb{R}^d)))
\end{aligned}$$

Natural to ask if the $\mathcal{M}^p$ spaces have some similar description?

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Banach Space Description of $\mathcal{M}^p$ spaces

Cannot extend p-nuclearity to $p > 1$. Instead consider p-summing operators $\Pi^p(X, Y)$, the operators T for which

$$\left(\sum_i \|Tx_i\|_Y^p \right) \leq c_T \sup_{\|x^*\|=1} \sum_i |x^*(x_i)|^p$$

for every finite sequence $x = \{x_i\}_{i=1}^N$ of elements in X.

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for every finite sequence $x = \{x_i\}_{i=1}^N$ of elements in X.Then using a Gabor operator frame, we can show that

$$\mathcal{M}^{p,q} \subset \Pi^q(M^{p'}(\mathbb{R}^d); M^q(\mathbb{R}^d)).$$

Don't have the reverse characterisation outside of the Banach-Gelfand triple!

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$\mathcal{M}^{p,q}$ spaces and Weyl symbols

In general we don't have a nice correspondence. But we can consider inclusion relations. Given $T \in \mathcal{M}^{p,1}$, the symbol of T is in $M^{1,p}(\mathbb{R}^{2d})$:

$$\begin{aligned}\|\sigma_T\|_{M^{1,p}} &= \|Q_S T(U^{-1}(w,z))\|_{L^{1,p}} \\ &\leq \sum_{n\in\mathbb{N}} |s_n| \cdot \left\| V_\varphi \phi_n\left(w + \frac{Jz}{2}\right) \overline{V_\varphi \psi_n\left(w - \frac{Jz}{2}\right)} \right\|_{L^{1,p}}\end{aligned}$$

Conversely, if $\sigma_T \in M^{\infty,q}(\mathbb{R}^{2d})$, then $T \in \mathcal{M}^{q,\infty}$.

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Conversely, if $\sigma_T \in M^{\infty,q}(\mathbb{R}^{2d})$, then $T \in \mathcal{M}^{q,\infty}$.By recognising $\mathcal{M}^{p,q}$ spaces as Banach–valued modulation spaces, we can use interpolation to show that if $p \leq q$, $T \in \mathcal{M}^{q,p} \implies \sigma_T \in M^{p,q}(\mathbb{R}^{2d})$. Conversly, for $q \leq p$, $\sigma_T \in M^{p,q}(\mathbb{R}^{2d}) \implies T \in \mathcal{M}^{q,p}$.

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Thank You

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