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Hagedorn states and the localization problem for Cohen's class

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The Wigner distribution

- Given $f \in L^2(\mathbb{R}^d)$ we define the Wigner distribution by
$$W(f)(z) = W(f)(x,\omega) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{f(x-\frac{t}{2})}e^{-2\pi i \langle t,\omega\rangle}\,dt.$$
- The cross-Wigner distribution is defined by $\int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i \langle t,\omega\rangle}\,dt.$
- Marginal properties: $\int_{\mathbb{R}^d} W(f)(x,\omega)\,dx = |\hat{f}(\omega)|^2, \quad \int_{\mathbb{R}^d} W(f)(x,\omega)\,d\omega = |f(x)|^2.$
- However, the Wigner function is in general *not positive!*

Cohen's class distributions

- Solution: Convolve with a nice function $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$.
- Cohen's class: $\left\{ Q(f) = W(f) * \sigma : \sigma \in \mathcal{S}'(\mathbb{R}^{2d}) \right\}$.
- Contains all weakly continuous, covariant quadratic time-frequency representations.

$$Q(\pi(z_0)f)(z) = Q(f)(z - z_0), \quad |Q(f,g)(0)| \leq \|f\|_2 \|g\|_2.$$

The localization problem for Cohen's class

- Given a set $\Omega \subset \mathbb{R}^{2d}$ and a Q in Cohen's class, find a signal in $L^2(\mathbb{R}^d)$ with $\|f\|_2 = 1$ that maximizes

$$\int_\Omega Q(f)\, dz.$$

- The choice $\sigma(z) = Amb(g)(z) = e^{\pi i \langle x,\omega \rangle} V_g g(z) = e^{\pi i \langle x,\omega \rangle} \int_{\mathbb{R}^d} g(t)\overline{g(t-x)} e^{-2\pi i \langle x,\omega \rangle}\, dz$, for $g \in L^2(\mathbb{R}^d)$ gives the *spectrogram*: $Q_\sigma(z) = |V_g f(z)|^2$.
- Spectrogram localization problem: Maximize

$$\int_\Omega |V_g f(z)|^2\, dz.$$

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Cohen's class from the QHA viewpoint

- Time-frequency shift: $\pi(z)f(t) = \pi(x,\omega)f(t) = f(t-x)e^{2\pi i\langle \omega, t\rangle}$.
- Parity operator: $Pf(t) = f(-t)$.
- Operator shift: $\alpha_z(S) = \pi(z)S\pi(z)^*$.
- Operator inversion $\check{S} = PSP$.
- Function-operator convolution: $f \star S = \int_{\mathbb{R}^{2d}} f(z)\alpha_z(S)\, dz$.
- Operator-operator convolution: $S \star T(z) = \mathrm{tr}(S\alpha_z(\check{T}))$.
- Fourier-Wigner transform: $\mathcal{F}_W(S)(z) = e^{-\pi i\langle x,\omega\rangle}\mathrm{tr}(\pi(-z)S)$.

Cohen's class as convolutions

Proposition (Luef-Skrettingland, 2019)

For $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$, its Cohen's class distribution is given by

$$Q_\phi(f) = (f \otimes f) \star L_\phi,$$

where $(f \otimes f)(g) = \langle g, f \rangle f$ and L_ϕ is the Weyl transform, the operator defined weakly by

$$\langle L_\phi f, g \rangle = \langle \phi, W(f, g) \rangle.$$

- So any operator S with Schwartz kernel defines a Cohen's class distribution via the formula

$$Q_S(f) = (f \otimes f) \star \check{S}.$$

A general solution

Lemma (Luef-Skrettingland, 2019)
Let $\Omega \subset \mathbb{R}^{2d}$ be measurable, and let S be an operator such that the operator $\chi_\Omega \star S$ is compact. Let $\lambda_1 \geq \lambda_2 \geq \dots$ denote its positive eigenvalues in decreasing order, and let f_i be the eigenfunction corresponding to λ_i. Then the eigenfunctions solve the localization problem in the sense that

$$\lambda_i = \int_\Omega Q_S(f_i)(z)\, dz = \max\left\{ \int_\Omega Q_S(f)(z)\, dz : f \perp \{f_1, f_2 \dots f_{i-1}\} \right\}.$$

- So we can solve the localization problem by solving the eigenvalue problem for the convolution $\chi_\Omega \star S$.

Another look at the spectrogram localization problem

- The case $S = g \otimes g$ gives $Q_S(f)(z) = |V_g f(z)|^2$.
- The corresponding convolution $\chi_\Omega \star (g \otimes g)$ gives a localization operator:

$$\chi_\Omega \star (g \otimes g) f = \int_\Omega V_g f(z) \pi(z) g \, dz = A_\Omega^g f.$$

Daubechies' theorem

Theorem (Daubechies, 1988)

Let g be the Gaussian $\phi_0 = 2^{1/4} e^{-\pi t^2}$ and D_R a disc of radius R centered at 0. The eigenfunctions of the localization operator $A_{D_R}^{\phi_0}$ are the Hermite functions:

$$\phi_n(t) = \frac{2^{1/4}}{\sqrt{n!}} \left(-\frac{1}{2\sqrt{\pi}} \right)^n e^{\pi t^2} \frac{d^n}{dt^n} e^{-2\pi t^2}.$$

The corresponding eigenvalue is

$$\lambda_n = \int_0^{\pi R^2} \frac{r^n}{n!} e^{-r} \, dr.$$

A first extension

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Theorem (Abreu-Gröchenig-Romero,2019,)

Let D_R be a disc of radius R centered at 0, and let $k \in \mathbb{N}_0$. The eigenfunctions of the localization operator $A_{D_R}^{\phi_k}$ are the Hermite functions: $\{\phi_n\}_{n\in\mathbb{N}_0}$ The corresponding eigenvalues are

$$\lambda_n = \frac{k!}{n!} \int_0^{\pi R^2} r^{n-k} e^{-r} \left(L_k^{n-k}(r) \right)^2 \, dr.$$

The Laguerre polynomials are given by

$$L_k^{\alpha}(t) = \sum_{j=0}^{k} (-1)^k \binom{k+\alpha}{k-j} \frac{t^j}{j!}.$$

Reinhardt domains

- The higher-dimensional condition on Ω is that χ_Ω is *polyradial*, that is

$$\chi_\Omega(x_1, x_2, \dots, x_d, \omega_1, \omega_2, \dots, \omega_d) = F_0(|z_1|, |z_2|, \dots, |z_d|).$$

- These are the Reinhardt domains.
- Ω is Reinhardt if there is a $W \subseteq \mathbb{R}_+^d$ such that

$$z \in \Omega \iff (|z_1|, |z_2|, \dots |z_d|) \in W.$$

- Ω is thin at infinity if for all $R > 0$,

$$\lim_{|z|\to\infty} |\Omega \cap B(z, R)| = 0.$$

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An unbounded set that is thin at infinity

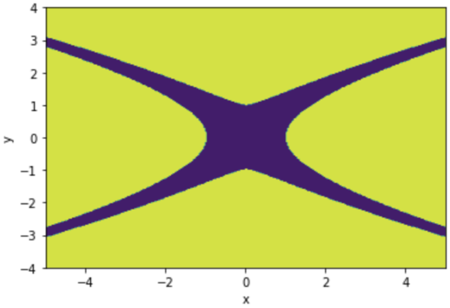
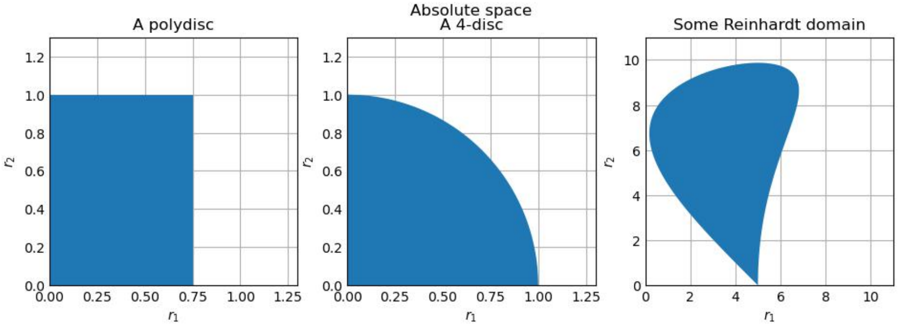


Figure: The set $\{(x,\omega) \in \mathbb{R}^2 \colon \left|x^{4/3} - y^2\right| < 1\}$.

Reinhardt domains in $\mathbb{R}^4$



Daubechies' theorem for Hermite functions, higher dimensions

Proposition

Let $\Omega \subset \mathbb{R}^{2d}$ be a Reinhardt domain with shadow W, and assume that Ω is thin at infinity. Let $k \in \mathbb{N}_0^d$. The eigenfunctions of the localization operator $A_\Omega^{\phi_k}$ are the Hermite functions, $\{\phi_n\}_{n \in \mathbb{N}_0^d}$ and the corresponding eigenvalues are

$$\lambda_{n,k}(W) = (2\pi)^d \frac{k!}{n!} \pi^{|n-k|} \int_W r^{2n-2k+1} e^{-\pi|r|^2} \left(\prod_{j=1}^{d} L_{k_j}^{n_j - k_j}(\pi r_j^2) \right)^2 \, dr.$$

Double orthogonality

- The eigenfunctions $\{f_n\}$ of a localization operator A_Ω^g can be characterized via *double orthogonality*, that is, the satisfaction of the two requirements

$$\int_{\mathbb{R}^{2d}} V_g f_n(z)\overline{V_g f_m(z)}\, dz = \delta_{n,m}.$$

and

$$\int_{\Omega} V_g f_n(z)\overline{V_g f_m(z)}\, dz = \lambda_n \delta_{n,m}.$$

- From Daubechies' theorem we have the double orthogonality relation for Hermite functions over Reinhardt domains:

$$\int_{\Omega} V_{\phi_k}\phi_n(z)\overline{V_{\phi_k}\phi_m(z)}\, dz = \lambda_{n,k}(W)\delta_{n,m}.$$

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Hagedorn wavepackets

- The ladder operators: $A_k^\dagger f(t) = \sqrt{\pi} t_k f(t) - \frac{1}{2\sqrt{\pi}} \frac{\partial f}{\partial t_k}(t)$ generate the Hermite functions via $\phi_n = \frac{1}{\sqrt{n!}} \left(A^\dagger\right)^n \phi_0$.
- Lagrangian frame: $Q, P \in \mathrm{Gl}(d, \mathbb{C})$ such that

$$Q^T P - P^T Q = 0, \quad Q^* P - P^* Q = 2i\mathrm{Id}_d.$$

- Equivalent to $T = \begin{pmatrix} \mathrm{Re}(Q) & \mathrm{Im}(Q) \\ \mathrm{Re}(P) & \mathrm{Im}(P) \end{pmatrix}$ being symplectic.
- Generalized Gaussian: $\phi_0[Q, P](t) = 2^{d/4} \det(Q)^{-1/2} e^{\pi i \langle t, PQ^{-1} t \rangle}$.

Hagedorn wavepackets

- Generalized ladder operators: $A^\dagger[Q,P] = \sqrt{\pi} i \left(P^* \cdot \mathrm{t} + Q^* \left(\frac{i\nabla}{2\pi} \right) \right)$.
- The k-th component acts by

$$A_k^\dagger[Q,P]f(t) = \sqrt{\pi} i \left(\sum_{l=1}^{d} \overline{p}_{lk} t_l f(t) + \frac{i}{2\pi} \overline{q}_{lk} \frac{\partial f}{\partial t_l}(t) \right).$$

- The Hagedorn wavepackets: $\phi_n[Q,P] = \frac{1}{\sqrt{n!}} \left(A^\dagger[Q,P] \right)^n \phi_0[Q,P]$.
- Inherits several properties of the Hermite functions: Orthonormality, three-term recurrence, Laguerre connection.
- $Q = \mathrm{Id}_d$, $P = i\mathrm{Id}_d$ gives the Hermite functions, Q, iP real gives rotated and dilated Hermite functions.

Hagedorn wavepackets in phase space

Lemma (Lasser-Troppmann, 2014,)

Let $k, n \in \mathbb{N}_0^d$, let $[Q, P]$ be a Lagrangian frame and T the corresponding symplectic matrix. We have

$$V_{\phi_k[Q,P]}\phi_n[Q,P](x,\omega) = e^{-\pi i\langle x,\omega\rangle} e^{-\frac{\pi}{2}|\zeta|^2} \prod_{j=1}^{d} \overline{H_{n_j,k_j}(\zeta_j)},$$

where $\zeta = \zeta(x,\omega) = -iP^T x + iQ^T\omega = T^{-1}\begin{pmatrix} x \\ \omega \end{pmatrix}$, and H_{n_j,k_j} is the corresponding complex Hermite polynomial.

- So $\left|V_{\phi_k[Q,P]}\phi_n[Q,P](x,\omega)\right| = \left|V_{\phi_k}\phi_n(T^{-1}(x,\omega))\right|$!

Daubechies' theorem for Hagedorn wavepackets

Proposition (S.)

Let Ω be a Reinhardt domain that is thin at infinity. Let $[Q,P]$ be a Lagrangian frame, T the corresponding symplectic matrix and let $k \in \mathbb{N}_0^d$.
The Hagedorn wavepackets $\{\phi_n[Q,P]\}_{n\in\mathbb{N}_0^d}$ satisfy the double orthogonality condition

$$\int_{T(\Omega)} V_{\phi_k[Q,P]}\phi_n[Q,P](z)\overline{V_{\phi_k[Q,P]}\phi_m[Q,P](z)}\, dz = \lambda_{n,k}(W)\delta_{n,m}.$$

The eigenvalues are

$$\lambda_{n,k}(W) = (2\pi)^d \frac{k!}{n!}\pi^{|n-k|}\int_W r^{2n-2k+1}e^{-\pi|r|^2}\left(\prod_{j=1}^d L_{k_j}^{n_j-k_j}(\pi r_j^2)\right)^2 dr.$$

Daubechies' theorem for Hagedorn wavepackets

Proposition (Cohen's class formulation) (S.)

The Hagedorn wavepackets $\{\phi_n[Q,P]\}_{n\in\mathbb{N}_0^d}$ are the successive maximizers of the localization problem

$$\max_{f\in L^2(\mathbb{R}^d),\|f\|_2=1} \int_{T(\Omega)} \left|V_{\phi_k[Q,P]} f(z)\right|^2 \, dz.$$

The successive maxima are

$$\lambda_{n,k}(W) = (2\pi)^d \frac{k!}{n!} \pi^{|n-k|} \int_W r^{2n-2k+1} e^{-\pi|r|^2} \left(\prod_{j=1}^{d} L_{k_j}^{n_j-k_j}(\pi r_j^2) \right)^2 \, dr.$$

Mixed-state localization operators with Hagedorn eigenfunctions

- Daubechies' theorem for pure states carries over to any S on the form $S = \sum_{n\in\mathbb{N}_0^d} \alpha_n \phi_n[Q,P] \otimes \phi_n[Q,P], \alpha \in \ell^1(\mathbb{N}_0^d)$.
- Hard condition to verify.
- Even if we can verify it, the eigenvalue expressions are in general not too nice.

Quantum double orthogonality

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Proposition (S.)

Let $\Omega \subset \mathbb{R}^{2d}$ be thin at infinity, and $S \in \mathcal{B}(L^2(\mathbb{R}^d))$ a trace class operator. The collection $\{\psi_n\}_{n\in\mathbb{N}_0^d}$ is the eigenfunctions of the mixed-state localization operator $\chi_\Omega \star S$ if and only if it is complete and satisfies the relations

$$\int_{\mathbb{R}^{2d}} Q_S\left(\psi_n, \psi_m\right)(z)\, dz = b_n \delta_{n,m}$$

and

$$\int_{\Omega} Q_S\left(\psi_n, \psi_m\right)(z)\, dz = \lambda_n \delta_{n,m}.$$

Examples

- $S = h \otimes h$: $\quad Q_{h\otimes h}(f,g)(z) = V_h f(z)\overline{V_h g(z)}$,

$$\int_\Omega V_h f(z)\overline{V_h g(z)}\, dz = \lambda_n \delta_{n,m}.$$

- $S \geq 0$: $\quad Q_S(f,g)(z) = \langle \sqrt{S}\pi(z)^* f, \sqrt{S}\pi(z)^* g\rangle$,

$$\int_\Omega \langle \sqrt{S}\pi(z)^* f, \sqrt{S}\pi(z)^* g\rangle\, dz = \lambda_n \delta_{n,m}.$$

- $S = 2^d P$: $\quad Q_{2^d P}(f,g) = W(f,g)(z)$,

$$\int_\Omega W(f,g)\, dz = \lambda_n \delta_{n,m}.$$

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Polyradial operators

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Definition

An operator $S : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is called *polyradial* if $\mathcal{F}_W(S)$ is a polyradial function. That is, there is a function $F_0 : \mathbb{R}_+^d \to \mathbb{C}$ such that

$$\mathcal{F}_W(S)(z) = F_0(|z_1|, |z_2|, \dots, |z_d|).$$

Examples

- $\phi_n \otimes \phi_n$: $\mathcal{F}_W(\phi_n \otimes \phi_n)(z) = \prod_{j=1}^d L_n^0(\pi|z_j|^2)e^{-\pi|z_j|^2}$.
- $2^d P$: $\mathcal{F}_W(2^d P)(z) = 1$.
- $Lf(t) = \frac{1}{2}\left(t^2 f(t) - \frac{f''(t)}{4\pi}\right)$: $\mathcal{F}_W(L)(z) = -\frac{1}{8\pi^2}\Delta\delta(z)$.
- Polyradial Hilbert-Schmidt operators $\cong L^2(\mathbb{R}_+^d)$.

Quantum double orthogonality for polyradial operators

Proposition (S.)

Let $S \in \mathcal{T}^1(L^2(\mathbb{R}^d))$ be polyradial. The Hermite functions $\{\phi_n\}_{n\in\mathbb{N}_0^d}$ are quantum doubly orthogonal on any Reinhardt domain Ω that is thin at infinity. That is, we have

$$\int_\Omega Q_S(\phi_n, \phi_m)(z)\, dz = \lambda_{n,S}(W)\delta_{n,m} = \int_\Omega \mathcal{F}_\sigma \mathcal{F}_W(S) * W(\phi_n)(z)\, dz\, \delta_{n,m},$$

where W is the Reinhardt shadow.

- So the Hermite functions are the eigenfunctions of $\chi_\Omega \star S$, and the eigenvalues are $\lambda_{n,S}(W)$.
- They are also the maximizers and maxima of

$$\max_{f\in L^2(\mathbb{R}^d), \|f\|_2=1} \int_\Omega Q_S(f)(z)\, dz.$$

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Hagedorn wavepackets as solutions

Corollary

Let $T \in \mathbb{R}^{2d \times 2d}$ be a symplectic matrix with block form $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Let $R \in \mathcal{T}^1(L^2(\mathbb{R}^d))$ be polyradial, and let S be a trace class operator with Weyl symbol $\mathcal{F}_\sigma \mathcal{F}_W(R)(T^{-1}z)$. If $Q = A + iB, P = C + iD$ then the Hagedorn wavepackets $\{\phi_n[Q,P]\}_{n \in \mathbb{N}_0^d}$ are quantum doubly orthogonal on any domain on the form $T(\Omega)$, where Ω is a Reinhardt domain. That is, we have

$$\int_{T(\Omega)} Q_S(\phi_n[Q,P], \phi_m[Q,P])(z)\, dz = \lambda_{n,R}(W)\delta_{n,m}.$$

Application: Gaussian Cohen's classes

- Let $M \in \mathbb{R}^{2d\times 2d}$ be positive and symmetric. The Gaussian

  $$g_M(z) = (2\pi)^{-d}\det(M^{-1})e^{-\frac{1}{2}\langle M^{-1}z,z\rangle}$$

  has a positive, trace class Weyl transform if and only if the matrix

  $$M + \frac{i}{4\pi}\begin{pmatrix} 0 & \mathrm{Id}_d \\ -\mathrm{Id}_d & 0 \end{pmatrix}$$

  is positive semidefinite.
- Among the admissible M, only those on the form $M = \begin{pmatrix} \tilde{M} & 0 \\ 0 & \tilde{M} \end{pmatrix}, M = \mathrm{diag}(m_1, m_2, \dots, m_d)$ are polyradial.

Gaussian Cohen's classes

- Williamson's theorem: $M = SKS^T = S \begin{pmatrix} \tilde{K} & 0 \\ 0 & \tilde{K} \end{pmatrix} S^T$, where $S \in \mathbb{R}^{2d \times 2d}$ is symplectic, and all k_i are positive.
- So for any admissible M we have $g_M(z) = g_K(S^{-1}z)$.
- Our main theorem holds for the mixed-state localization operator $\chi_{S(\Omega)} \star L_{g_M}$!
- The eigenfunctions are $\{\phi_n[Q,P]\}_{n \in \mathbb{N}_0^d}$, where $Q = A + iB$, $P = C + iD$, and the eigenvalues are

$$\int_\Omega g_K \star W(\phi_n)(z)\, dz.$$

Gaussian Cohen's classes

Corollary

Let $M = S^T K S \in \mathbb{R}^{2d \times 2d}$ be admissible, and let $\Omega \subset \mathbb{R}^{2d}$ be a Reinhardt domain that is thin at infinity. Then the Hagedorn wavepackets are the successive maximizers of the localization problem

$$\max_{f \in L^2(\mathbb{R}^d), \|f\|_2 = 1} \int_{S(\Omega)} (g_M * W(f))(z)\, dz.$$

The successive maxima are

$$\lambda_n = \int_\Omega (g_K * W(\phi_n))(z)\, dz.$$

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Thank you!

References I

[1] Luís Daniel Abreu, Karlheinz Gröchenig, and José Luis Romero. "Harmonic analysis in phase space and finite Weyl-Heisenberg ensembles". In: *J. Stat. Phys.* 174 (2019).

[2] Ingrid Daubechies. "Time-frequency localization operators: a geometric phase space approach". In: *IEEE Trans. Inform. Theory* 34.4 (1988).

[3] Caroline Lasser and Stephanie Troppmann. "Hagedorn wavepackets in time-frequency and phase space". In: *J. Fourier Anal. Appl.* 20.4 (2014), pp. 679–714.

[4] Elliott H. Lieb and Yaron Ostrover. "Localization of multidimensional Wigner distributions". In: *J. Math. Phys.* 51.10 (2010).

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References II

[5] Franz Luef and Eirik Skrettingland. "Mixed-state localization operators: Cohen's class and trace class operators". In: *J. Fourier Anal. Appl.* 25.4 (2019).

[6] Jayakumar Ramanathan and Pankaj Topiwala. "Time-frequency localization via the Weyl correspondence". In: *SIAM J. Math. Anal.* 24.5 (1993).

[7] Erling Svela. *Reinhardt domains, Hagedorn wavepackets and mixed-state localization operators*. In preparation.